Infinity Groupoids as Models for Homotopy Types

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Contents

Objectives

- Study topological categories as a model of ∞-categories.

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- Prove that Moore path categories are a model of the fundamental ∞-groupoid of a topological space.
- Prove that the coherent nerve of an ∞-groupoid is equivalent to the classical nerve of the associated topological category.
- Assess whether the model of Moore path categories can help to interpret results from homotopy type theory.

Contents

Higher categories

- Historically, there have been many definitions for ∞-categories, and each one is considered a *model* of higher homotopy.
  - Globular models (Batanin, Berger, etc.).
  - Quasi-categories (Joyal, Lurie).
  - Topologically enriched categories (Bergner, Lurie, etc.).

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- An ∞*-groupoid* is an ∞-category whose n-morphisms are invertible up to $(n+1)$-morphisms, for all $n \geq 1$.
- Grothendieck's *homotopy hypothesis* states that, for each topological space X, the *fundamental* ∞*-groupoid* $\Pi_\infty(X)$ encodes the homotopical structure of X.

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- A topological category $\mathcal{C}$ is an ∞-*groupoid* if $\mathrm{h}\mathcal{C}$ is a groupoid.

Homotopy hypothesis

Top **Top-Cat**

Homotopy hypothesis

$$\mathbf{Top} \underset{\mathrm{Sing}}{\overset{|\cdot|}{\rightleftarrows}} \mathbf{sSet}_Q \qquad\qquad \mathbf{Top\text{-}Cat}$$

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$$\mathbf{Top} \underset{\mathrm{Sing}}{\overset{|\cdot|}{\rightleftarrows}} \mathbf{sSet}_Q \underset{k^!}{\overset{k_!}{\rightleftarrows}} \mathbf{sSet}_J \underset{\mathrm{N}^{\mathfrak{R}}}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathbf{sSet\text{-}Cat} \underset{\mathrm{Sing}_e}{\overset{|\cdot|_e}{\rightleftarrows}} \mathbf{Top\text{-}Cat}$$

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$$\mathbf{Top} \underset{\mathrm{Sing}}{\overset{|\cdot|}{\leftrightarrows}} \mathbf{sSet}_Q \underset{k^! \circ \mathrm{N}^{\mathfrak{R}} \circ \mathrm{Sing}_e}{\overset{|\cdot|_e \circ \mathfrak{C} \circ k_!}{\rightleftarrows}} \infty\text{-}\mathbf{Grpd}.$$

Nerve and realization: Homotopy coherent nerve

There is a *cosimplicial object* defined for each $[n] \in \Delta$ as the simplicial category $\Delta^{\Re}[n]$ with:

- $\mathrm{Obj}(\Delta^{\Re}[n]) = [n] = \{0, \ldots, n\}$
- For every $i, j \in \mathrm{Obj}(\Delta^{\Re}[n])$, $\mathrm{Hom}(i, j) = (\Delta[1])^{(j-i-1)}$

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The *simplicial path* $\mathfrak{C} : \textbf{sSet} \to \textbf{sSet-Cat}$ is defined for every $X \in \textbf{sSet}$ as

$$\mathfrak{C}(X) = \int^{[n]\in\Delta} X_n \otimes \Delta^{\mathfrak{R}}[n].$$

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$$P_{x,y}^M X = \{(f, r) \in X^{\mathbb{R}_+} \times \mathbb{R}_+ \mid f(0) = x \text{ and } f(s) = y\ \forall s \geq r\}.$$

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- The composition is defined by

$$\begin{aligned} \circ : \ P_{x,y}^M X \times P_{y,z}^M X &\longrightarrow \quad P_{x,z}^M X \\ ((f,r),(g,s)) &\longmapsto (f * g, r+s) \end{aligned}$$

$$(f * g)(t) = \begin{cases} f(t) & \text{if } 0 \leq t < r \\ g(t-r) & \text{if } t \geq r \end{cases}$$

The fundamental ∞-groupoid as a Moore path category

Let $\Omega^M_x(X)$ be the group-like topological monoid defined as $P^M_{x,x}X$. The *delooping* functor $\mathbb{D} : \mathbf{tMon} \to \mathbf{Top\text{-}Cat}_0$ sends $M \in \mathbf{tMon}$ to the topological category with one object $*$ and $\mathrm{Hom}(*, *) = M$.

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Main Theorem

Let (X, x) be a path-connected well-pointed topological space. The topological space $|\,\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathbb{D}\,\Omega^M_x(X)))|$ is a classifying space for $\Omega^M_x(X)$ and, as a consequence,

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Let $\Omega_x^M(X)$ be the group-like topological monoid defined as $P_{x,x}^M X$. The *delooping* functor $\mathbb{D} : \mathbf{tMon} \to \mathbf{Top\text{-}Cat}_0$ sends $M \in \mathbf{tMon}$ to the topological category with one object $*$ and $\mathrm{Hom}(*, *) = M$.

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Hence, the ∞-groupoid $\Pi_\infty^M(X)$ is weakly homotopy equivalent to the ∞-groupoid $(|\cdot|_e \circ \mathfrak{C} \circ k_! \circ \mathrm{Sing})(X)$.

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The *topological nerve* $\mathrm{N}^t : \mathbf{Top\text{-}Cat}_0 \to \mathbf{sTop}_0$ is the functor that sends $\mathbb{D}\, M \in \mathbf{Top\text{-}Cat}_0$ with $\mathrm{Hom}(*, *) = M$ to the simplicial set with $\mathrm{N}^t_0(\mathbb{D}\, M) = *$ and $\mathrm{N}^t_n(\mathbb{D}\, M) = M^n$.

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Milgram defined a functorial classifying space $\mathrm{B}(M)$ for every topological group-like monoid M, which is equivalent to

$$\mathrm{B}(M) = |\,\mathrm{N}^t(\mathbb{D}\, M)|_t.$$

Observations about the Main Theorem

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Let (X, x) be a path-connected pointed topological space. Then, there is a natural weak homotopy equivalence

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Let (X, x) be a path-connected pointed topological space. Then, there is a natural weak homotopy equivalence

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It is enough to show that, for any topological space X,

$$|\,\mathsf{N}^{\mathfrak{R}}(\mathsf{Sing}_e(\mathbb{D}\,\Omega_x^M X))| \simeq \mathsf{B}(\Omega_x^M X) = |\,\mathsf{N}^t(\mathbb{D}\,\Omega_x^M X)|_t.$$

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Diagonal simplicial nerve

There is a cosimplicial object defined for each $[n] \in \Delta$ as the simplicial category $\Delta^d[n]$ with:

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The *diagonal simplicial nerve* $\mathrm{N}^d : \textbf{sSet-Cat} \to \textbf{sSet}$ is the functor that sends any simplicial category $\mathcal{C}$ to

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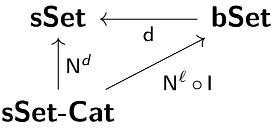
and can be factorized as



Idea of the proof I

To prove the main theorem, it is enough to prove

$$|\,\mathsf{N}^{\mathfrak{R}}(\mathsf{Sing}_e(\mathbb{D}\,M))| \overset{?}{\simeq} |\,\mathsf{N}^t(\mathbb{D}\,M)|_t$$

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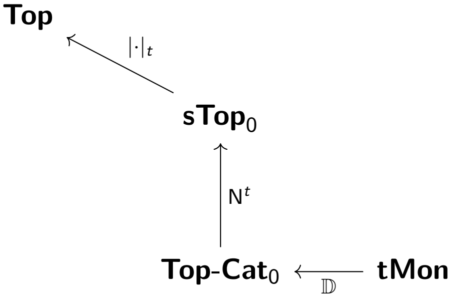

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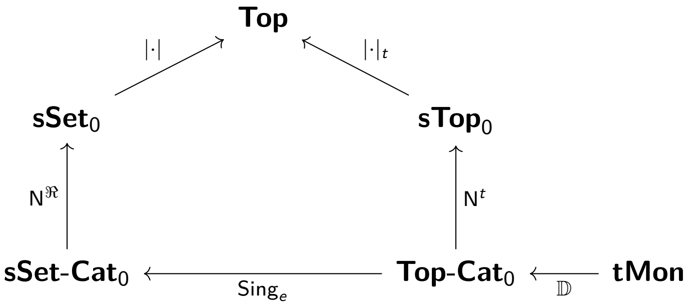

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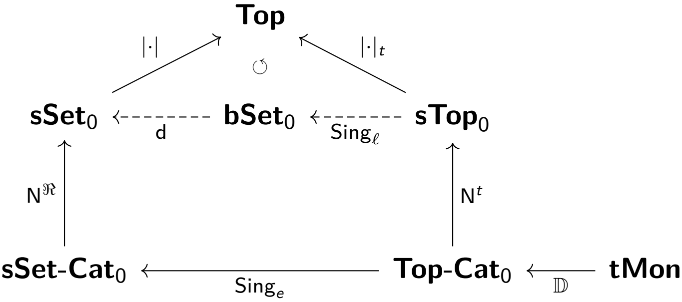

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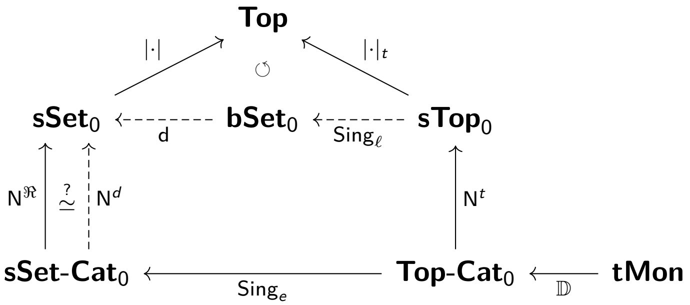

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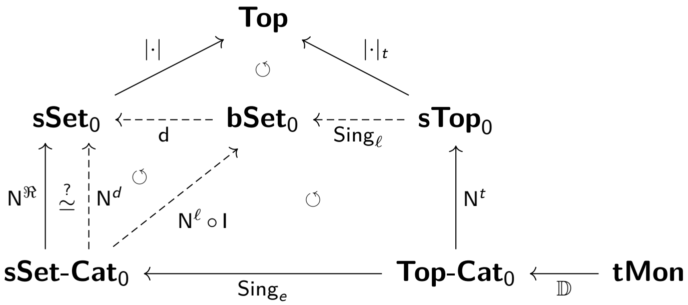

Idea of the proof II

Goal: For every topological group-like monoid M,

$$\mathrm{N}^d(\mathrm{Sing}_e(\mathbb{D}\, M)) \overset{?}{\simeq} \mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathbb{D}\, M)).$$

Idea of the proof II

Goal: For every ∞-groupoid $\mathcal{C}$,

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Idea of the proof II

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Goal: For every fibrant simplicial category $\mathcal{C}$ with $\mathrm{h}\mathcal{C}$ a groupoid,

$$\mathrm{N}^d(\mathcal{C}) \overset{?}{\simeq} \mathrm{N}^{\mathfrak{R}}(\mathcal{C}).$$

We can divide this statement into two subgoals:

- Proving that, for any strict simplicial groupoid $\mathcal{G}$,

$$\mathrm{N}^d(\mathcal{G}) \overset{?}{\simeq} \mathrm{N}^{\mathfrak{R}}(\mathcal{G}).$$

- Using simplicial localization to transfer this result to weak simplicial groupoids, i.e., any fibrant simplicial category $\mathcal{C}$ with $\mathrm{h}\mathcal{C}$ a groupoid.

Total simplicial nerve

There is a cosimplicial object defined for each $[n] \in \Delta$ as the simplicial category $\Delta^T[n]$ with:

- $\mathrm{Obj}(\Delta^T[n]) = [n]$.
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The *total simplicial nerve* $\mathsf{N}^T : \mathbf{sSet\text{-}Cat} \to \mathbf{sSet}$ is defined for every $\mathcal{C} \in \mathbf{sSet\text{-}Cat}$ as

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Theorem (Hinich 2015)

For any strict simplicial groupoid $\mathcal{G}$, $\mathsf{N}^d(\mathcal{G}) \simeq \mathsf{N}^T(\mathcal{G}) \simeq \mathsf{N}^{\mathfrak{R}}(\mathcal{G})$.

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Homotopy type theory embraces the homotopical interpretation of type theory, adding the univalence axiom and higher inductive types.

Any result developed inside homotopy type theory can be formalized and checked using computer software.

Dependent type theory

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$$\vdash A \text{ type} \qquad \vdash a : A \qquad \vdash a = b \qquad \vdash A = B$$

Dependent type theory

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$$\vdash A \text{ type} \qquad \vdash a : A \qquad \vdash a = b \qquad \vdash A = B$$

Any judgement can have type dependency. For example, a family B indexed by A is the judgement $x : A \vdash B(x)$ type.

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Many types resemble common mathematical constructions, for example functions $A \to B$, products $A \times B$, sum type $A + B$, and the natural numbers $\mathbf{N}$.

Identity types

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We can consider identity types of identity types, and so on recursively, which creates a higher dimensional structure for every type with a weakly associative composition and a weak inverse.

Interpretation

Category theory	Type theory
Fibrant object A	Type declaration A type
Fibration $B \to A$	Dependent family $x : A \vdash B(x)$ type
Global section $1 \to A$	Term $x : A$
Product $A \times B$	Product $A \times B$
Coproduct $A \sqcup B$	Sum $A + B$
Exponential object A^B	Function $A \to B$
Path object $\mathrm{Path}(A) \to A \times A$	Identity type $a, b : A \vdash \mathrm{Id}_A(a, b)$

Higher inductive types

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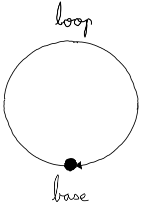
- base : $\mathbf{S}^1$
- loop : $\mathsf{Id}_{\mathbf{S}^1}(\mathsf{base}, \mathsf{base})$



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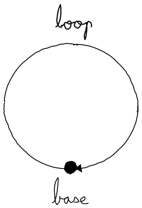
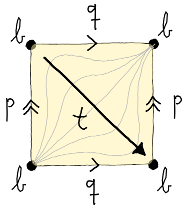


- b : $\mathbf{T}^2$
- p : $\mathsf{Id}_{\mathbf{T}^2}(b, b)$
- q : $\mathsf{Id}_{\mathbf{T}^2}(b, b)$
- t : $\mathsf{Id}_{\mathsf{Id}_{\mathbf{T}^2}(b,b)}(p \cdot q, q \cdot p)$



Using Moore path categories

$$
\begin{array}{cc}
\Pi^M_\infty(\widetilde{A}) & A \\
\coprod_{\bar{x},\bar{y}\in\widetilde{A}} \Pi^M_\infty(P^M_{\bar{x},\bar{y}}\widetilde{A}) & \sum_{x,y:A} \mathsf{Id}(x,y) \\
\coprod_{\bar{x},\bar{y}\in\widetilde{A}} \coprod_{\bar{p}_1,\bar{q}_1\in P^M_{\bar{x},\bar{y}}\widetilde{A}} \Pi^M_\infty(P^M_{\bar{p}_1,\bar{q}_1}(P^M_{\bar{x},\bar{y}}\widetilde{A})) & \sum_{x,y:A} \sum_{p_1,q_1:\mathsf{Id}(x,y)} \mathsf{Id}(p_1,q_1) \\
\vdots & \vdots
\end{array}
$$

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\vdots & \vdots
\end{array}
$$

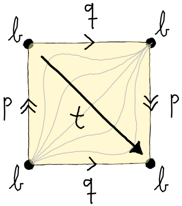
The interpretation of the type-theoretic circle and the type-theoretic torus have the same homotopy types as the fundamental ∞-groupoids of the circle and the torus.

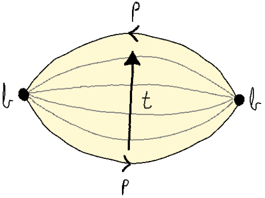
Future research

Further research is needed for studying other cases like the Klein bottle or the real projective spaces:

- b : **K**.
- p : $\mathsf{Id}_{\mathbf{K}}(b, b)$
- q : $\mathsf{Id}_{\mathbf{K}}(b, b)$
- t : $\mathsf{Id}_{\mathsf{Id}_{\mathbf{K}}(b,b)}(p \cdot q, q \cdot p^{-1})$

- b : $\mathbf{RP}^2$
- p : $\mathsf{Id}_{\mathbf{RP}^2}(b, b)$
- t : $\mathsf{Id}_{\mathsf{Id}_{\mathbf{RP}^2}(b,b)}(p, p^{-1})$





Infinity Groupoids as Models for Homotopy Types

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