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# Infinity groupoids as models for homotopy types

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# Abstract

In higher category theory, $\infty$-groupoids are $\infty$-categories whose morphisms are weakly invertible at all orders. Every topological space has an associated $\infty$-groupoid, named its fundamental $\infty$-groupoid, which encodes the information of higher paths over the space. The statement that every space can be recovered up to homotopy from its fundamental $\infty$-groupoid is known as Grothendieck's homotopy hypothesis. In this thesis, we choose a model of $\infty$-categories based on topologically enriched categories, and discuss the homotopy hypothesis in this context, as well as a model of the fundamental $\infty$-groupoid based on Moore path categories.

The core result in our work is that the coherent nerve of an $\infty$-groupoid is equivalent to the classical nerve of the associated topologically enriched category. We provide a detailed proof of this fact since we have not found it in the literature.

As an application, we assess whether the model of the fundamental $\infty$-groupoid based on Moore path categories is useful in the higher-categorical interpretation of homotopy type theory, a field of mathematical logic which relates Martin-Löf's type theory with the study of $\infty$-groupoids. Homotopy type theory allows the formalization of homotopical results in computer proof assistants.

# Contents

# Introduction

In higher category theory, one considers categories with not only morphisms between objects, but generally $n$-morphisms between $(n-1)$-morphisms for all $n \geq 1$. Historically, there have been many definitions for $\infty$-categories, and each one is considered a *model* of higher homotopy. The higher categories studied in this thesis are the $\infty$-groupoids, whose $n$-morphisms are "weakly invertible" for all $n \geq 1$. The most popular model for higher categories is based on simplicial sets satisfying a weak Kan condition. It was pioneered by Joyal and developed by Lurie [Lur09]. In this work, we analyze an alternative model studied by Lurie [Lur09] and Amrani [Amr13], which is based on topologically enriched categories.

In fact, higher category theory and homotopy theory are closely related. For each topological space $X$, we can build an $\infty$-groupoid called *fundamental $\infty$-groupoid* $\Pi_\infty(X)$, which encodes the homotopical structure of the higher paths over $X$. More specifically, $\Pi_\infty(X)$ has as objects the points of $X$, as 1-morphisms the paths on $X$, as 2-morphisms the homotopies between paths, and so on recursively. The idea of the fundamental $\infty$-groupoid was originally drafted by Grothendieck, who thought that the study of $\infty$-groupoids should be "equivalent" to the study of homotopy types of topological spaces. This statement became known as the *homotopy hypothesis*. Indeed, it needs to be reformulated for each model of $\infty$-groupoids, and it is a tautology in some cases, so it cannot be treated as a conjecture.

The homotopy hypothesis depends on a suitable definition of equivalence between $\infty$-groupoids and topological spaces. Suppose that we have a model structure over topological spaces and another one over some category of $\infty$-groupoids. Then, it turns out that the notion of a zigzag of Quillen equivalences between model categories is well suited to represent such an "equivalence", because it proves that the homotopical structures induced by the model categories are equivalent. In the first chapter, we review the language of model categories, adjunctions and Quillen equivalences. We also recap the model structures over topological spaces and simplicial sets, and a Quillen equivalence between these categories. In addition, we develop a generalization of the usual adjunction between topological spaces and simplicial sets, called the *nerve and realization* pattern. This pattern allows us to define an adjunction from simplicial sets to other categories using a suitable cosimplicial object, and it will be used to define all the adjunctions used in the rest of this work.

In the second chapter, we prove that topologically enriched categories offer a convenient model for $\infty$-groupoids, thanks to its inherent strict composition. This will be accomplished by proving the homotopy hypothesis associated with this model. In the literature, it is well-known that there is a Quillen equivalence between simplicial sets and simplicially enriched categories [Lur09], through a functor named *homotopy coherent nerve* and defined using the nerve and realization pattern. The origins of this functor go back to work of Boardman and Vogt [BV73], and Cordier [Cor82]. The proof of the homotopy hypothesis associated to topologically enriched categories extends the Quillen equivalence generated by the homotopy coherent nerve. Our main references have been [Amr11] and [McG20].

The final section of the second chapter tackles the problem of computing the fundamental $\infty$-groupoid of a topological space, by presenting Moore path categories. For every topological space $X$ and every two points $x, y \in X$, we can consider the set of all paths between $x$ and $y$, called path space $P_{x,y}X$. This space has a weakly associative composition, weak unit and weak inverses. But we can make them "strict" by considering the homotopically equivalent Moore path space:

$$P^M_{x,y}X = \{(f, r) \in X^{\mathbb{R}_+} \times \mathbb{R}_+ \mid f(0) = x \text{ and } f(s) = y \ \forall s \geq r\}.$$

The Moore path space has strict associative composition and strict unit, but weak inverses. In particular, when taking the same points as source and target, the Moore path space $P^M_{x,x}X$ is a topological monoid, which is usually denoted by $\Omega^M_x X$.

The *Moore path category* of a space $X$ is a topologically enriched category with objects the points of $X$ and as homsets the Moore path spaces. The name of Moore path category was pioneered by Brown [Bro09]. The Bachelor's thesis of McGarry [McG20] ends by proving that the Moore path category models the fundamental $\infty$-groupoid. In this thesis, we provide a different proof. The proof from McGarry depends on a proposition of an article by Rivera and Zeinalian [RZ18], which itself depends on a claim without reference about an alternative classifying space of $\Omega^M_x X$ based on the homotopy coherent nerve. Because we have not found any reference in the literature proving this claim, we decided to write our own proof, which is developed in the third chapter. Using this claim, we finish the second chapter by presenting a direct proof of the fact that the Moore path category models the fundamental $\infty$-groupoid.

A classifying space $\mathrm{B}(G)$ of a topological group $G$ is defined as the quotient of a weakly contractible space $\mathrm{E}(G)$ by a proper free action of $G$. Any classifying space of a group has the universal property that for every topological space $X$ there is a bijection between homotopy classes of maps $X \to \mathrm{B}(G)$ and isomorphism classes of $G$-bundles over $X$. The third chapter is devoted to reviewing the theory of classifying spaces, and then proving the existence of an alternative classifying space of $\Omega^M_x X$ based on the homotopy coherent nerve. The proof presented here is novel work, but uses ideas from the literature. In particular, the last part of the argument was inspired on an article by Hinich [Hin07].

The last chapter of this thesis puts into perspective the model of the fundamental $\infty$-groupoid as a Moore path category in the field of homotopy type theory. Homotopy type theory [Uni13] is a refinement of Martin-Löf type theory based on the interpretation of types as $\infty$-groupoids [AW09; BG10; Lum09]. Any result developed inside homotopy type theory can be formalized and checked using computer software. The choice of a suitable model of $\infty$-groupoids is essential to translate theories formalized inside type theory to homotopy theoretic results. One of the innovations of homotopy type theory is the introduction of higher inductive types as a tool for freely generating the higher structure of a type by a set of generators. In particular, higher inductive types can be used to generate types with a higher structure inspired by the fundamental $\infty$-groupoid of some topological space. With this technique, we can obtain type theoretic versions of common finite CW-complexes such as the circle or the torus. However, confirming that the resulting higher inductive types actually realize the fundamental $\infty$-groupoid of the original topological space is still an open problem. In this last chapter, using the explicit structure of Moore path categories and the interpretation of homotopy type theory in model categories by van den Berg and Garner [BG12], we show that the type theoretic circle and torus actually correspond to the $\infty$-groupoids associated with the topological version of these structures. An extension of this result to other finite CW-complexes is a future project.

# Chapter 1

# Preliminaries about model categories

This chapter is devoted to summarizing well-known concepts about model categories. Subsequent chapters will use these ideas as foundations. The first section introduces model categories, and presents some basic properties. In the second one, adjunctions and Quillen equivalences are presented and discussed. In the third one, we introduce a model structure over the category of topological spaces, and several classical homotopical constructions. Finally, we present simplicial sets as a tool for approaching the higher categorical contents of the subsequent chapters. During the rest of this work, basic knowledge about category theory will be assumed. For details of categorical concepts, see [Lan78].

## 1.1 Model categories

In this section, we review a modern definition of model categories, following [Hov07]. The concept appeared originally in works of Quillen [Qui67]. To understand the definition, we first need to set out some preliminary concepts.

An object $A$ of $\mathcal{C}$ is a *retract* of another object $B$ if there are morphisms $i : A \to B$ and $r : B \to A$ such that $r \circ i = \mathrm{Id}_A$. On the other hand, a morphism $f : A \to B$ is a *retract* of another $g : C \to D$ if and only if there is a commutative diagram of the form

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & A \\ \big\downarrow f & & \big\downarrow g & & \big\downarrow f \\ B & \longrightarrow & D & \longrightarrow & B \end{array}$$

where the horizontal composites are identities.

Let $\mathcal{C}$ be any category, $f : A \to B$ and $g : C \to D$ two morphisms from $\mathcal{C}$. We say that $f$ has the *left lifting property* (LLP) with respect to $g$, or equivalently that $g$ has the *right lifting property* (RLP) with respect to $f$, if for every pair of morphisms $u : A \to C$ and $v : B \to D$ such that $g \circ u = v \circ f$, there exists a morphism $\alpha : B \to C$ making the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ \big\downarrow f & \overset{\alpha}{\nearrow} & \big\downarrow g \\ B & \xrightarrow{v} & D \end{array}$$

The last preliminary is an addition to the original definition of Quillen, which has become a convention in modern texts. A *functorial factorization* on $\mathcal{C}$ is a map denoted $(E, \alpha, \beta)$ from every morphism $f : A \to B$ to an object $E(f)$ and a pair of composable morphisms $\alpha(f) : A \to E(f)$ and $\beta(f) : E(f) \to B$ such that $f = \beta(f) \circ \alpha(f)$ and for every other morphism $g : C \to D$ with a commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{v} & D \end{array}$$

there exists a morphism $H(u, v)$ natural in both variables such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle \alpha(f)}\downarrow & & \downarrow{\scriptstyle \alpha(g)} \\ E(f) & \xrightarrow{H(u,v)} & E(g) \\ {\scriptstyle \beta(f)}\downarrow & & \downarrow{\scriptstyle \beta(g)} \\ B & \xrightarrow{v} & D \end{array}$$

**Definition 1.1.1.** A *model structure* over a category $\mathcal{C}$ is composed of three distinguished classes of morphisms: *weak equivalences* ($\xrightarrow{\sim}$), *fibrations* ($\twoheadrightarrow$), and *cofibrations* ($\hookrightarrow$), such that the following conditions hold:

1. All three classes contain all isomorphisms.
2. All three classes are closed under composition and under retracts.
3. The class of weak equivalences satisfies the *two-out-of-three* property: for any pair of morphisms $f : A \to B$ and $g : B \to C$ such that two of $f$, $g$ or $g \circ f$ are weak equivalences, then the third is also a weak equivalence.
4. Any *trivial cofibration* (a cofibration which is also a weak equivalence) has the LLP with respect to any fibration. Conversely, any *trivial fibration* (a fibration which is also a weak equivalence) has the RLP with respect to any cofibration.
5. There exist two functorial factorizations $(E, \alpha, \beta)$ and $(F, \gamma, \delta)$ such that for any morphism $f : A \to B$, $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration and $\delta(f)$ is a fibration.

Then, a *model category* is a complete and cocomplete category $\mathcal{C}$ equipped with a model structure.

Let $\mathcal{C}$ be a model category. Because $\mathcal{C}$ is complete and cocomplete, in particular it always has an initial object $\varnothing$ and a terminal one $*$. Then, an object $A$ of $\mathcal{C}$ is *fibrant* if the unique morphism $A \to *$ is a fibration. Dually, an object $A$ is *cofibrant* if the unique morphism $\varnothing \to A$ is a cofibration. Furthermore, we will denote by $\mathcal{C}_f$, $\mathcal{C}_c$ and $\mathcal{C}_{cf}$ the full subcategories of fibrant objects, cofibrant objects, and fibrant and cofibrant objects, respectively.

Applying the first functorial factorization $(E, \alpha, \beta)$ to the unique morphism $\iota_A : \varnothing \to A$ we obtain a cofibration $\alpha(\iota_A) : \varnothing \to E(\iota_A)$, and a trivial fibration $\beta(\iota_A) : E(\iota_A) \to A$. Then, there exists a *cofibrant replacement functor* $A \mapsto \mathrm{Q}A$ defined by $\mathrm{Q}A := E(\iota_A)$ such that $\mathrm{Q}A$ is a cofibrant object and there is a natural transformation $q_A := \beta(\iota_A) : \mathrm{Q}A \to A$ which is a trivial fibration. Dually, we can use $(F, \gamma, \delta)$ and $\tau_A : A \to *$ to define a *fibrant replacement functor* $A \mapsto \mathrm{R}A$ such that $\mathrm{R}A := F(\tau_A)$ is fibrant and there is a natural transformation $r_A := \gamma(\tau_A) : A \to \mathrm{R}A$ which is a trivial cofibration.

**Remark 1.1.2.** Let $A$ be a fibrant object of a model category, with its associated fibration $\tau_A : A \to *$. Applying the cofibrant replacement we obtain $\mathrm{Q}A$ which is cofibrant and has a trivial fibration $q_A : \mathrm{Q}A \to A$. Then, the composition of fibrations $\tau_A \circ q_A : \mathrm{Q}A \to *$ must be a fibration by the axioms of the model structure. Therefore, applying the cofibrant replacement to a fibrant object produces an object which is fibrant and cofibrant. The dual of this result follows by the same argument.

Let $A$ be an object in a model category $\mathcal{C}$, and $A \amalg A$ (resp. $A \times A$) be the coproduct (resp. product) of $A$ with itself. Then, consider the *fold map* as the morphism $A \amalg A \to A$, defined with the universal property of the coproduct applied to the identity $\mathrm{Id}_A$. Dually, define the *diagonal* as the morphism $A \to A \times A$, defined with the universal property of the product applied to the identity $\mathrm{Id}_A$.

**Definition 1.1.3.** Let $\mathcal{C}$ be a model category, and $f, g : A \to B$ two morphisms in $\mathcal{C}$.

(i) A *cylinder object* $\mathrm{Cyl}(A)$ for $A$ is a factorization of the fold map of $A$ into a cofibration $i_0 + i_1 : A \amalg A \to \mathrm{Cyl}(A)$ followed by a weak equivalence $\mathrm{Cyl}(A) \to A$.

(ii) A *path object* $\mathrm{Path}(B)$ for $B$ is a factorization of the diagonal map of $B$ into a weak equivalence $B \to \mathrm{Path}(B)$ followed by a fibration $(p_0, p_1) : \mathrm{Path}(B) \to B \times B$.

(iii) A *left homotopy* from $f$ to $g$ is a map $H : \mathrm{Cyl}(A) \to B$ for some cylinder object $\mathrm{Cyl}(A)$ for $A$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$. Furthermore, $f$ and $g$ are *left homotopic*, denoted $f \overset{l}{\sim} g$, if there is a left homotopy from $f$ to $g$.

(iv) A *right homotopy* from $f$ to $g$ is a morphism $K : A \to \mathrm{Path}(B)$ for some path object $\mathrm{Path}(B)$ for $B$ such that $p_0 \circ K = f$ and $p_1 \circ K = g$. Furthermore, $f$ and $g$ are *right homotopic*, denoted $f \overset{r}{\sim} g$, if there is a right homotopy from $f$ to $g$.

(v) The morphisms $f$ and $g$ are *homotopic*, denoted $f \sim g$, if they are both left and right homotopic. Furthermore, $f$ is a *homotopy equivalence* if there is a map $h : B \to A$ such that $h \circ f \sim \mathrm{Id}_A$ and $f \circ h \sim \mathrm{Id}_B$.

Observe that due to the functorial factorizations, we can obtain a functorial cylinder (resp. path) object as a factorization of the fold (resp. diagonal) map. Also, when $A$ is cofibrant and $B$ is fibrant, the left and right homotopy relations coincide in $\mathcal{C}(A, B)$ and are equivalence relations (see the Whitehead Theorem [GJ09, Theorem 1.10]). Thanks to this fact, the following construction is well-defined:

**Definition 1.1.4.** Let $\mathcal{C}$ be a model category. Define the *homotopy category* $\mathbf{Ho}(\mathcal{C})$ as the category with:

- The objects of $\mathbf{Ho}(\mathcal{C})$ are the ones from $\mathcal{C}$ that are both fibrant and cofibrant.
- For any two objects $A$ and $B$ of $\mathbf{Ho}(\mathcal{C})$, define $\mathbf{Ho}(\mathcal{C})(A, B) := [A, B]$, where the brackets denote the homotopy classes of $\mathcal{C}(A, B)$.
- For any morphisms $[f] \in \mathbf{Ho}(\mathcal{C})(A, B)$ and $[g] \in \mathbf{Ho}(\mathcal{C})(B, C)$, the composition is defined as $[g] \circ [f] := [g \circ f] \in \mathbf{Ho}(\mathcal{C})(A, C)$.
- For every object $A \in \mathbf{Ho}(\mathcal{C})$, the identity element is $[\mathrm{Id}_A] \in \mathbf{Ho}(\mathcal{C})(A, A)$.

The homotopy category has another equivalent definition which only depends on the class of weak equivalences. Define the *localization* $\mathcal{C}[W^{-1}]$ of a category $\mathcal{C}$ at a class of morphisms $W$ as the formal inversion of all the morphisms of $W$. In general, this definition of localization has some set-theoretical difficulties: if $W$ is a proper class, the localization can be a "large" category, in the sense of not being locally small. In the case of $W$ being the class of weak equivalences, it can be proven that there is an equivalence of categories between $\mathbf{Ho}(\mathcal{C})$ and $\mathcal{C}[W^{-1}]$ (for details, see [Hov07, Section 1.2]). This equivalence proves that $\mathcal{C}[W^{-1}]$ is well-defined as a locally small category when taking $\mathcal{C}$ to be a model category and $W$ the weak equivalences.

## 1.2 Adjunctions and Quillen equivalences

In this section, a short introduction to the well-known theory of adjunctions and Quillen equivalences is offered. The proofs and details of the results about adjunctions can be found with modern notation in [Rie16].

**Definition 1.2.1.** An *adjunction* between two categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that there is a natural bijection $\mathcal{D}(FA, B) \cong \mathcal{C}(A, GB)$ for every $A \in \mathcal{C}$ and $B \in \mathcal{D}$. We call $F$ the *left adjoint* to $G$ and $G$ the *right adjoint* to $F$, and also denote the adjunction by the following notation:

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G.$$

**Proposition 1.2.2.** *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction. Then, there are:*

(i) *A natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ called unit of the adjunction, whose component $\eta_A : A \to GFA$ at $A \in \mathcal{C}$ is defined as the image of the identity morphism $\mathrm{Id}_{FA}$ through the natural bijection of the adjunction.*

(ii) *A natural transformation $\epsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ called counit of the adjunction, whose component $\epsilon_A : FGA \to A$ at $A \in \mathcal{D}$ is defined as the image of the identity morphism $\mathrm{Id}_{GA}$ through the natural bijection of the adjunction.*

**Proposition 1.2.3.** *A pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ is an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ if and only if there exists a pair of natural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ such that for every $A \in \mathcal{C}$ and $B \in \mathcal{D}$, the following diagrams commute:*

$$\begin{array}{ccc} FA & \xrightarrow{F\eta_A} & FGFA \\ & {\scriptstyle \mathrm{Id}_{FA}} \searrow & \downarrow {\scriptstyle \epsilon_{FA}} \\ & & FA \end{array} \qquad \begin{array}{ccc} GB & \xrightarrow{\eta_{GB}} & GFGB \\ & {\scriptstyle \mathrm{Id}_{GB}} \searrow & \downarrow {\scriptstyle G\epsilon_B} \\ & & GB \end{array}$$

**Proposition 1.2.4.** *Every left adjoint functor preserves colimits and every right adjoint functor preserves limits.*

Given model categories, we can impose further conditions on an adjunction to ensure that it respects the model structures. This leads to the following definition:

**Definition 1.2.5.** Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction between model categories. Then we say that the pair of $F$ and $G$ is a *Quillen adjunction* between $\mathcal{C}$ and $\mathcal{D}$ if any of the following equivalent conditions is satisfied:

(i) $F$ preserves cofibrations and trivial cofibrations.

(ii) $G$ preserves fibrations and trivial fibrations.

(iii) $F$ preserves cofibrations and $G$ preserves fibrations.

(iv) $F$ preserves trivial cofibrations and $G$ preserves trivial fibrations.

The equivalences between those conditions follow directly from the axioms of the model structures and the definition of adjunction (for details, see [Hir09, Proposition 8.5.3]). As in the case of adjunctions, we call $F$ a *left Quillen functor* and $G$ a *right Quillen functor*.

**Remark 1.2.6.** Observe that since the composition of natural isomorphisms is a natural isomorphism, the composition of left (resp. right) adjoint functors is left (resp. right) adjoint. The same is true with Quillen adjunctions: the composition of left (resp. right) Quillen adjoint functors is left (resp. right) Quillen adjoint.

A Quillen adjunction also preserves fibrant (resp. cofibrant) objects and weak equivalences under certain conditions. The first property follows from the well-known Ken Brown's Lemma [Hov07, Lemma 1.1.12]:

**Proposition 1.2.7.** *Every left Quillen functor preserves weak equivalences between cofibrant objects, and every right Quillen functor preserves weak equivalences between fibrant objects.*

**Proposition 1.2.8.** *Every left Quillen functor preserves cofibrant objects, and every right Quillen functor adjoint preserves fibrant objects.*

*Proof.* Consider a left Quillen functor $F : \mathcal{C} \to \mathcal{D}$ and an initial object $\varnothing$ of $\mathcal{C}$. Because $F$ is a left adjoint and $\varnothing$ is a colimit, by Proposition 1.2.4 we know that $F\varnothing$ is also an initial object. On the other hand, for every cofibrant object $A$ of $\mathcal{C}$ there is a cofibration $\varnothing \hookrightarrow A$. Therefore, because $F$ is a left Quillen functor, it preserves cofibrations, and $F\varnothing \to FA$ is a cofibration, which implies that $FA$ is a cofibrant object of $\mathcal{D}$. Dually, the same argument proves that every right Quillen functor preserves fibrant objects. □

In addition to the previous properties, every right or left Quillen functor induces a functor between homotopy categories. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction. The *total left derived functor* $\mathbb{L}F : \mathbf{Ho}(\mathcal{C}) \to \mathbf{Ho}(\mathcal{D})$ is the functor induced by the composite of $F \circ \mathrm{R}$ restricted to $\mathcal{C}_{cf}$. Similarly, the *total right derived functor* $\mathbb{R}G : \mathbf{Ho}(\mathcal{D}) \to \mathbf{Ho}(\mathcal{C})$ is induced by the composite $G \circ \mathrm{Q}$ restricted to $\mathcal{D}_{cf}$. Furthermore, these functors form an adjunction $\mathbb{L}F : \mathbf{Ho}(\mathcal{C}) \rightleftarrows \mathbf{Ho}(\mathcal{D}) : \mathbb{R}G$ between homotopy categories. The details about why this is well defined and the adjunction induced at the level of homotopy can be found in [Lur09, Subsection A.2.5].

Finally, we can introduce Quillen equivalences as the Quillen adjunctions which induce an equivalence of categories between the homotopy categories. We introduce also an equivalent condition which is also commonly used as definition for Quillen equivalences. The details about the equivalence between the conditions can be found in [Hov07, Proposition 1.3.13].

**Definition 1.2.9.** A Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a *Quillen equivalence* if any of the following equivalent conditions is satisfied:

(i) For all cofibrant $A$ in $\mathcal{C}$ and fibrant $B$ in $\mathcal{D}$, a morphism $FA \to B$ is a weak equivalence in $\mathcal{D}$ if and only if the adjunct morphism $A \to GB$ is a weak equivalence in $\mathcal{C}$.

(ii) $F$ and $G$ induce an equivalence between the homotopy categories.

**Remark 1.2.10.** Consider two composable left (resp. right) Quillen adjunctions $F_1$ and $F_2$. Thanks to the facts discussed in Remark 1.2.6, $F_2 \circ F_1$ is also a left (resp. right) Quillen adjunction. Then, if two out of three of $F_1$, $F_2$ and $F_2 \circ F_1$ are Quillen equivalences, so is the third. This fact follows directly from the definition, because weak equivalences have the two-out-of-three property.

Recall that a functor $F$ is said to *reflect weak equivalences* if for every morphism $f$, when $Ff$ is a weak equivalence so is $f$. In some cases, proving that a Quillen adjunction satisfies the definition of a Quillen equivalence can be a challenging task. For this reason, the following proposition (proven in [Hov07, Corollary 1.3.16]) will be very useful:

**Proposition 1.2.11.** *The following are equivalent:*

(i) *The pair of $F$ and $G$ is a Quillen equivalence.*

(ii) *$F$ reflects weak equivalences and for all fibrant objects $B \in \mathcal{D}$ the adjunction counit $\epsilon_B : FGB \to B$ is a weak equivalence.*

(iii) *$G$ reflects weak equivalences and for all cofibrant objects $A \in \mathcal{C}$ the adjunction unit $\eta_A : A \to GFA$ is a weak equivalence.*

## 1.3 Topological spaces

The goal of this section is to set some notation regarding topological spaces, and to review the construction of the Quillen model structure over topological spaces.

Let **Top** denote the category of topological spaces and continuous functions. This category of topological spaces lacks many good categorical properties. It is complete, cocomplete and cartesian monoidal using the function space $Y^X$ of continuous maps from $X$ to $Y$ (with the compact-open topology [Mun17, Section 7.5]) as internal homset functor. But **Top** is not cartesian closed, and the product does not commute with colimits in general. In subsequent sections we will need to use some of these properties; therefore we need to consider a “nicer” category of topological spaces.

A topological space $X$ is *weakly Hausdorff* if, for every map $f : K \to X$ where $K$ is compact Hausdorff, $f(K)$ is closed in $X$. On the other hand, a subset $U$ of $X$ is *compactly open* if for every continuous map $f : K \to X$ where $K$ is compact Hausdorff, $f^{-1}(U)$ is open in $K$. Similarly, $U$ is *compactly closed* if for every such map $f$, $f^{-1}(U)$ is closed in $K$. Then, a topological space $X$ is a *Kelley space* if every compactly open subset is open, or equivalently, if every compactly closed subset is closed. A Kelley space that is also weakly Hausdorff is called a *compactly generated space*. Denote the full subcategory of **Top** consisting of the compactly generated spaces by **CGTop**.

By [Hov07, Proposition 2.4.22], the category **CGTop** is cartesian closed, and therefore solves the drawbacks of **Top** commented earlier. Additionally, by [Hov07, Corollary 2.4.24], the model structure that we will define over **Top** is Quillen equivalent to the corresponding one over **CGTop** via the inclusion functor. Hence, we will not lose any homotopical information by restricting our category of topological spaces to **CGTop**. From now on, we will work exclusively in **CGTop**, which will be denoted by **Top** for the sake of simplicity.

Consider the $n$-dimensional sphere $\mathbb{S}^n$, with base-point $(1, 0, \ldots, 0) \in \mathbb{S}^n$. Then, for any pointed topological space $(X, x)$ and any $n \geq 1$, we define the *$n$-homotopy group* $\pi_n(X, x)$ as the group of homotopy classes of base-point preserving maps from $\mathbb{S}^n$ to $X$. For the case $n = 0$, the set of homotopy classes of base-point preserving maps from $\mathbb{S}^0$ to $X$ is a set instead of a group, called the set of *path components*.

**Definition 1.3.1.** A map of topological spaces $f : X \to Y$ is a *weak homotopy equivalence* if $f$ induces a bijection between path components and for any $n \in \mathbb{N} \setminus \{0\}$ and any $x \in X$, it also induces an isomorphism on the homotopy groups:

$$\pi_n(f, x) : \pi_n(X, x) \xrightarrow{\cong} \pi_n(Y, f(x)).$$

Let $I := [0, 1]$ be the closed unit interval, and $\mathbb{D}^n$ the $n$-dimensional disk of radius 1 centered at the origin, both with the Euclidean topology. Recall that a CW-complex is a topological space $X$ constructed inductively in the following way:

- Select a set $X_0$ as the 0-cells of $X$.
- Inductively, form the $n$-skeleton $X_n$ from $X_{n-1}$ by attaching $n$-cells $\{e_i^n\}_{0 \leq i \leq m}$ by maps $\{\varphi_i^n : \mathbb{S}^{n-1} \to X_{n-1}\}_{0 \leq i \leq m}$. Each cell is attached by means of a pushout:

$$\begin{array}{ccc} \mathbb{S}^{n-1} & \xrightarrow{\varphi_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \mathbb{D}^n & \longrightarrow & \mathbb{D}^n \coprod_{\varphi_i^n} X_{n-1} \end{array}$$

Then, if $X$ has finite dimension $n$, define $X = X_n$, else define $X = \bigcup_n X$, with the weak topology [Hat19, p. 519].

**Definition 1.3.2.** A map of topological spaces $p : E \to B$ is a *Serre fibration* if it satisfies the homotopy lifting property with respect to any CW-complex, i.e., $p$ has the RLP with respect to $(\mathrm{Id}, 0) : A \to A \times I$ for every CW-complex $A$.

As proven originally by Quillen, the category of topological spaces admits a model structure, usually called Quillen model structure. In this work, we consider the same model structure but applied to compactly generated spaces. For a proof of the existence of this model structure over compactly generated spaces, see [Hov07, Theorem 2.4.19].

**Theorem 1.3.3.** [Qui67, Section II]. *There is a model structure over* **Top***, usually referred to as the Quillen model structure, defined as follows:*

- *The weak equivalences are the weak homotopy equivalences.*
- *The fibrations are the Serre fibrations.*
- *The cofibrations are the maps with the LLP with respect to the trivial fibrations.*

*Furthermore, every topological space is fibrant, and the retracts of CW-complexes are cofibrant.*

Using these definitions, we can relate more classical homotopical constructions to their associated concepts in model categories. For example, for every space $X$ there is a canonical choice of cylinder object as the product $X \times I$ and of path object as the mapping space $X^I$. Additionally, the left homotopy from the model structure using this cylinder object coincides with the topological definition of homotopy.

On the other hand, **Top** admits other model structures such as the Hurewicz model structure. Although we will not use this model structure, we will need the definition of its cofibrations at some point:

**Definition 1.3.4.** A map of topological spaces $i : A \to X$ is a *Hurewicz cofibration* if it satisfies the homotopy extension property, i.e., $i$ has the left lifting property with respect to $p_1 : X^I \to X$ for every topological space $X$.

## 1.4 Simplicial sets

Simplicial sets are a common tool for studying topological spaces from a combinatorial point of view. The relation between simplicial sets and topological spaces is modelled by a Quillen equivalence, which proves that the two categories have the same homotopical information.

In this section we will start with a recap of basic definitions related to simplicial sets. Afterwards, we will introduce a less known generalization of the adjunction between simplicial sets and topological spaces, called *nerve and realization adjunction.* Finally, we will review the Quillen equivalence between simplicial sets and topological spaces.

**Definition 1.4.1.** Define the *simplex category* $\Delta$ as the category with objects the linearly ordered sets $[n] := \{0, 1, \ldots, n\}$ for all $n \geq 0$, and morphisms all set functions $[n] \to [m]$ which are non-decreasing.

**Definition 1.4.2.** A *simplicial object* in a category $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$. Dually, a *cosimplicial object* in $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$. Because functors form a category with natural transformations, there is a category of simplicial objects in $\mathcal{C}$, and one of cosimplicial objects in $\mathcal{C}$.

**Definition 1.4.3.** A *simplicial set* is a simplicial object in **Set**. The category of simplicial sets is denoted by $\mathbf{sSet} := \mathrm{Func}(\Delta^{op}, \mathbf{Set})$.

Any simplicial set $X$ has a set for each $[n]$, denoted $X[n]$ or $X_n$. On the other hand, in the simplex category there are two special types of morphisms: the *injections* $\delta_i^n : [n-1] \to [n]$ and the *surjections* $\sigma_i^n : [n+1] \to [n]$, both defined for every $0 \leq i \leq n$ by

$$\delta_i^n(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases} \qquad \sigma_i^n(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

Because any simplicial set $X$ is a functor, these morphisms induce functions, called *faces* $d_i^n := X(\delta_i^n) : X_n \to X_{n-1}$ and *degeneracies* $s_i^n := X(\sigma_i^n) : X_n \to X_{n+1}$. Furthermore, every morphism in the simplex category can be expressed as a composition of a surjection and an injection. Therefore, the sets $\{X_n\}_{n \in \mathbb{N}}$ with the faces and degeneracies determine a unique simplicial set.

Also, from the properties of the injections and surjections, we can derive the following *simplicial identities*, that all simplicial set have to satisfy:

$$\begin{aligned} d_i^{n-1} \circ d_j^n &= d_{j-1}^{n-1} \circ d_i^n \qquad \text{if } i < j \\ d_i^{n+1} \circ s_j^n &= \begin{cases} s_{j-1}^{n-1} \circ d_i^n & \text{if } i < j \\ \mathrm{Id}_{X_n} & \text{if } i = j \text{ or } i = j+1 \\ s_j^{n-1} \circ d_{i-1}^n & \text{if } i > j+1 \end{cases} \\ s_i^{n+1} \circ s_j^n &= s_{j+1}^{n+1} \circ s_i^n \qquad \text{if } i \leq j \end{aligned}$$

The *standard n-simplex* is the simplicial set defined by $\Delta[n] := \Delta(\cdot, [n])$. By the Yoneda lemma [Rie16, Theorem 2.2.4], for each simplicial set $X$ and each $n \in \mathbb{N}$, we have

$$X_n \cong \mathbf{sSet}(\Delta(\,\cdot\,, [n]), X) = \mathbf{sSet}(\Delta[n], X).$$

Then, the *boundary* of $\Delta[n]$, denoted $\partial\Delta[n]$, is defined as the smallest sub-simplicial-set of $\Delta[n]$ containing $\Delta[n]_0, \Delta[n]_1, \ldots, \Delta[n]_{n-1}$. The *k-th horn* $\Lambda^k[n]$ is the sub-simplicial-set of $\Delta[n]$ obtained from removing the $k$-th face. The horns with $0 < k < n$ are usually called *inner horns*, and the ones with $k = 0$ or $k = n$ are the *outer horns.*

### 1.4.1 Nerve and realization

In the subsequent chapters, we will define several adjunctions from simplicial sets to other categories. It turns out that all these adjunctions follow a general pattern, which arises as a generalization of the adjunction between simplicial sets and topological spaces. The general pattern was originally found by Kan in [Kan58], but it has been used in more modern contexts like [Lur21, Variant 1.1.7.7] and [Hin07].

Let $\mathcal{C}$ be any cocomplete category and $Q : \Delta \to \mathcal{C}$ be a cosimplicial object in $\mathcal{C}$. For any object $A \in \mathcal{C}$, we can apply the contravariant homset functor, obtaining a set $\mathcal{C}(Q[n], A)$ for every $[n] \in \Delta$. Since $Q$ is a cosimplicial object, the generating morphisms of $\Delta$, $\delta^n_j : [n-1] \to [n]$ and $\sigma^n_j : [n+1] \to [n]$, induce morphisms $Q(\delta^n_j) : Q[n-1] \to Q[n]$ and $Q(\sigma^n_j) : Q[n+1] \to Q[n]$ satisfying the standard cosimplicial identities. Hence, because $\mathcal{C}(\,\cdot\,, A)$ is a contravariant functor, $\mathcal{C}(Q[\,\cdot\,], A))$ is a simplicial set. Therefore, the following functors are well-defined:

**Definition 1.4.4.** Define the *$Q$-nerve* as the functor $\mathrm{N}^Q : \mathcal{C} \to \mathbf{sSet}$ which maps an object $A \in \mathcal{C}$ to the simplicial set defined for every $[n] \in \Delta$ by

$$\mathrm{N}^Q_n(A) = \mathcal{C}(Q[n], A).$$

Now assume also that $\mathcal{C}$ is cocomplete. Then, it has a canonical *copowering* functor over **Set**, i.e., there exists a functor $\otimes : \mathbf{Set} \times \mathcal{C} \to \mathcal{C}$ defined by

$$S \otimes B := \coprod_{s \in S} B.$$

This copowering over **Set** is required to define the candidate adjoint functor to the $Q$-nerve:

**Definition 1.4.5.** Define the *$Q$-realization* functor $|\cdot|_Q : \mathbf{sSet} \to \mathcal{C}$ as the left Kan extension of $\mathrm{N}^Q$, i.e., for all $X \in \mathbf{sSet}$:

$$|X|_Q = \int^{[n]\in\Delta} X_n \otimes Q[n].$$

As proved originally by [Kan58], the pair of $Q$-nerve and $Q$-realization always form an adjunction. The proof given bellow follows the original argument but with a more modern point of view:

**Proposition 1.4.6.** *The $Q$-nerve and $Q$-realization form an adjunction*

$$|\cdot|_Q : \mathbf{sSet} \rightleftarrows \mathcal{C} : \mathrm{N}^Q .$$

*Proof.* By the following chain of isomorphisms, the functors form an adjunction:

$$\begin{aligned}
\mathcal{C}(|X|_Q, A) &= \mathcal{C}\left(\left(\int^{[n]\in\Delta} X_n \otimes Q[n]\right), A\right) \\
&\cong \int_{[n]\in\Delta} \mathcal{C}(X_n \otimes Q[n], A) && (1.1)\\
&\cong \int_{[n]\in\Delta} \mathbf{Set}(X_n, \mathcal{C}(Q[n], A)) && (1.2)\\
&= \int_{[n]\in\Delta} \mathbf{Set}(X_n, \mathrm{N}^Q_n(A)) \\
&\cong \mathbf{sSet}(X, \mathrm{N}^Q(A)). && (1.3)
\end{aligned}$$

The isomorphism of (1.1) follows from the Hom functor sending colimits in the first argument to limits. The one from (1.2) follows as a natural isomorphism of the canonical copowering over **Set**. Using again that the Hom functor sends colimits in the first argument to limits, we obtain the desired isomorphism:

$$\mathcal{C}(X_n \otimes Q[n], A) = \mathcal{C}\left(\coprod_{x\in X_n} Q[n], A\right) \cong \prod_{x\in X_n} \mathcal{C}(Q[n], A) \cong \mathbf{Set}(X_n, \mathcal{C}(Q[n], A)).$$

Finally, the isomorphism from (1.3) follows directly from the definition of maps between simplicial sets. □

As said before, we will use this pattern to define several functors through this work. In the next subsection, the classic singular simplicial and geometric realization adjunction will be presented as a pair of $Q$-nerve and $Q$-realization. In fact, the name of $Q$-realization comes from the geometric realization.

On the other hand, the name of the $Q$-nerve comes from the other uses of this pattern, where the right adjoint is some type of nerve. For example, consider the category of categories **Cat**. There is a trivial cosimplicial object given by the elements of $\Delta$, the categories $[n]$. Those cosimplicial objects induce a $[\,\cdot\,]$-nerve, which coincides with the traditional nerve. For any category $C$ and any $n \in \mathbb{N}$, it is defined as

$$\mathrm{N}_n(C) := \mathrm{N}_n^{[\,\cdot\,]}(C) = \mathbf{Cat}([n], C).$$

### 1.4.2 Quillen equivalence with topological spaces

In this subsection we want to sum up the well-known Quillen equivalence between simplicial sets and topological spaces. The underlying adjunction between these categories will be presented using the nerve and realization pattern introduced in the last subsection. Consider the following cosimplicial object:

**Definition 1.4.7.** The *standard topological $n$-simplex* is the topological space

$$\Delta^n := \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid 0 \le t_i \le 1, \textstyle\sum_{i=0}^n t_i = 1\}.$$

Define the *standard topological simplex* map as $\Delta^\bullet : [n] \mapsto \Delta^n$.

**Proposition 1.4.8.** *$\Delta^\bullet$ is a cosimplicial object in* **Top**.

*Proof.* For any map $f : [n] \to [m]$, the image $\Delta f : \Delta^n \to \Delta^m$ is defined for all $(t_0, \ldots, t_n)$ as

$$\Delta f(t_0, \ldots, t_n) = \left(\textstyle\sum_{f(i)=0} t_i, \sum_{f(i)=1} t_i, \ldots, \sum_{f(i)=m} t_i\right)$$

and has all the desired simplicial identities. □

Following the construction from Definitions 1.4.4 and 1.4.5, the cosimplicial objects $\Delta^\bullet$ induce a definition of a nerve and a realization:

**Definition 1.4.9.** The *singular simplicial set* $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$ is the functor which, for any $X \in \mathbf{Top}$,

$$\mathrm{Sing}_n(X) := \mathrm{N}_n^{\Delta^\bullet} = \mathbf{sSet\text{-}Cat}(\Delta^n, X).$$

On the other hand, we define the *geometric realization* $|\cdot| : \mathbf{sSet} \to \mathbf{Top}$ as the functor which, for any $X \in \mathbf{sSet}$,

$$|X| := |X|_{\Delta^\bullet} = \int^{[n]\in\Delta} X_n \otimes \Delta^n.$$

By Proposition 1.4.6, these functors form an adjunction: $|\cdot| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \mathrm{Sing}$. The next step is proving that this adjunction is in fact a Quillen equivalence. Before, we need to define the Quillen model structure over **sSet**:

**Definition 1.4.10.** A morphism of simplicial sets $f : X \to Y$ is called a *weak homotopy equivalence* if $|f| : |X| \to |Y|$ is a weak homotopy equivalence of topological spaces.

**Definition 1.4.11.** A morphism of simplicial sets $f : X \to Y$ is a *Kan fibration* if it has the RLP with respect to the horn inclusions $\Lambda^k[n] \to \Delta[n]$ for all $n \geq 1$ and all $0 \leq k \leq n$.

**Definition 1.4.12.** A simplicial set $X$ is called a *Kan complex* if for any $n \geq 1$ and $0 \leq k \leq n$, every map $\Lambda^k[n] \to X$ admits an extension $\Delta[n] \to X$ through the canonical inclusion $\Lambda^k[n] \to \Delta[n]$.

**Theorem 1.4.13.** [Qui67, Section II.3]. *The category of simplicial sets admits a model structure named Quillen model structure, in which:*

- *The weak equivalences are the weak homotopy equivalences.*
- *The fibrations are the Kan fibrations.*
- *The cofibrations are the injective maps.*

*Furthermore, all objects are cofibrant, and the fibrant objects are the Kan complexes.*

As in the case of topological spaces, there are other model structures over simplicial sets. In the following chapters we will study one of them. In particular, this alternative model has the following fibrant objects:

**Definition 1.4.14.** A simplicial set $X$ is called a *quasi-category* if for any $n \geq 1$ and $0 < k < n$, every map $\Lambda^k[n] \to X$ admits an extension $\Delta[n] \to X$ through the canonical inclusion $\Lambda^k[n] \to \Delta[n]$.

Now that we have defined model structures for **Top** and **sSet**, we are ready to check that the previous adjunction between them is in fact a Quillen adjunction. A proof about this result in a modern context can be found in [Hov07, Theorem 3.6.7].

**Theorem 1.4.15.** *The adjunction* $|\cdot| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \mathrm{Sing}$ *is a Quillen adjunction.*

Because it is a left Quillen functor, it preserves colimits and cofibrations. But we can also prove that it preserves finite limits and fibrations. The preservation of finite limits only works when choosing a "nice" category of spaces, like the category of compactly generated spaces that we are using. The details can be found in [GJ09, Proposition 2.4, Theorem 10.10].

**Proposition 1.4.16.** *The geometric realization preserves finite limits.*

**Theorem 1.4.17.** *The geometric realization of a Kan fibration is a Serre fibration.*

Finally, we can show that the unit and the counit of this adjunction are in fact weak equivalences, and derive the existence of a Quillen equivalence. The proof of the unit can be found in [GJ09, Proposition 11.1], and the one from the counit in [Hov07, Theorem 3.6.7].

**Proposition 1.4.18.** *For any simplicial set $X$, the unit map $\eta_X : X \to \mathrm{Sing}(|X|)$ is a weak equivalence of simplicial sets.*

**Proposition 1.4.19.** *For every topological space $X$, the counit map $\epsilon_X : |\mathrm{Sing}(X)| \to X$ is a weak homotopy equivalence of topological spaces.*

**Theorem 1.4.20.** *The Quillen adjunction* $|\cdot| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \mathrm{Sing}$ *is a Quillen equivalence.*

*Proof.* By Proposition 1.2.11, it is enough to prove that the geometric realization reflects weak equivalences and that the counit of the adjunction is a weak equivalence for all fibrant objects. The first fact follows from the definition of the geometric realization, and the second one corresponds to Proposition 1.4.19. □

# Chapter 2

# The fundamental $\infty$-groupoid as a Moore path category

As discussed in the Introduction, there are several models of $\infty$-groupoids. Once a model is chosen, the homotopy hypothesis is a theorem proving that the model category of $\infty$-groupoids has a zigzag of Quillen equivalences ending at the model category of topological spaces. In particular, for each topological space $X$ there is an associated $\infty$-groupoid named the fundamental $\infty$-groupoid $\Pi_\infty(X)$ which models the structure of higher paths over $X$.

The first section of this chapter introduces a model for higher categories based on topological categories. In the second one, simplicial categories will be presented, as a tool to relate simplicial sets and topological categories. In the third, we will prove the homotopy hypothesis for this model of $\infty$-groupoids. Finally, the last section will be used to introduce a more manageable model to the fundamental $\infty$-groupoid. This proposed model is based on Moore path categories as an alternative to the topological category constructed from a topological space in the previous section.

## 2.1 Topological categories

In this section we present a notion of categories with higher morphisms based on enriched categories. An enriched category has homsets with some extra structure, making the homsets be objects in a selected category. We will consider two examples of this construction: simplicial categories, which are categories enriched in simplicial sets, and topological categories, which are enriched in topological spaces.

**Definition 2.1.1.** Let $\mathcal{M}$ be a monoidal category (see [Lur09, Appendix A.1.3 and A.1.4]) with product $\times$ and unit $I$. Define an $\mathcal{M}$-*enriched category* $\mathcal{C}$ as:

- A collection of objects.
- For every pair of objects $X, Y \in \mathcal{C}$, the morphisms between $X$ and $Y$ are an object of $\mathcal{M}$ denoted $\mathcal{C}(X,Y)$ or $\mathrm{Hom}(X,Y)$.
- For every triple of objects $X, Y, Z \in \mathcal{C}$, an associative composition map

$$\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z).$$

- For every object $X \in \mathcal{C}$, a morphism $I \to \mathcal{C}(X,X)$ of $\mathcal{M}$ which represents the identity element.

An *enriched functor* is a functor $F : \mathcal{C} \to \mathcal{D}$ between enriched categories over the same category $\mathcal{M}$, which consists of a map between the objects of $\mathcal{C}$ and $\mathcal{D}$ and a map between each collection of morphisms

$$\mathcal{C}(X, Y) \longrightarrow \mathcal{D}(FX, FY)$$

which must preserve the composition and identity elements. Any functor between enriched categories will be assumed to be an enriched functor. Then, we can consider the category of all enriched categories over a category $\mathcal{M}$, with morphisms being the enriched functors.

As we mentioned earlier, the category of compactly generated topological spaces **Top** is a monoidal category [Hov07, Proposition 2.4.22]. Hence, we can consider enriched categories over the category of compactly generated topological spaces:

**Definition 2.1.2.** A *topological category* is a category enriched over the category **Top**. Furthermore, we will denote the category of all topological categories as **Top-Cat**.

Our next goal is to present a model structure in the category of all topological categories. This model structure depends on choosing a suitable notion of "weak equivalence" between two topological categories. First, we need to define a homotopy category associated to each topological category, without the need of a model structure:

**Definition 2.1.3.** Let $\mathcal{C}$ be a topological category. Define the *homotopy category* $\mathrm{h}\mathcal{C}$ as the category with:

- The objects of $\mathrm{h}\mathcal{C}$ are the objects of $\mathcal{C}$.
- For any $X, Y \in \mathcal{C}$, define $\mathrm{h}\mathcal{C}(X, Y) = \pi_0\mathcal{C}(X, Y)$.
- The composition and identity are induced from $\mathcal{C}$ by applying the functor $\pi_0$.

**Theorem 2.1.4.** [Amr13, Theorem 1.1]. *The category* **Top-Cat** *has a model structure with:*

- *The weak equivalences are the functors $F : \mathcal{C} \to \mathcal{D}$ such that:*
  - *For any $X, Y \in \mathcal{C}$, the map $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ is a weak equivalence in* **Top**.
  - *The induced morphism $\mathrm{h}F : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is an equivalence of categories.*
- *The fibrations are the functors $F : \mathcal{C} \to \mathcal{D}$ such that:*
  - *For any $X, Y \in \mathcal{C}$, the map $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ is a fibration in* **Top**.
  - *For every $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, and every weak equivalence $e : FX \to Y$ in $\mathcal{D}$, there exists an object $Z \in \mathcal{C}$ with a weak equivalence $d : X \to Z$ in $\mathcal{C}$ such that $Fd = e$.*
- *The cofibrations are the maps with the LLP with respect to the trivial fibrations.*

*Furthermore, all objects of* **Top-Cat** *are fibrant, and the cofibrant objects are the topological categories in which each mapping space is a cofibrant space.*

On the other hand, we can select a subcategory of topological categories which we want to associate with topological spaces. Those topological categories will be the ones with all higher morphisms being invertible "up to homotopy".

**Definition 2.1.5.** A topological category $\mathcal{C}$ is an $\infty$-*groupoid* if $\mathrm{h}\mathcal{C}$ is a groupoid. The subcategory of $\infty$-groupoids will be denoted $\infty$-**Grpd**.

The intuitive property of invertibility "up to homotopy" of higher morphisms can be interpreted using the condition on the homotopy category. Consider any $\infty$-groupoid $\mathcal{C}$ and a morphism $f : X \to Y$ of $\mathcal{C}$. Because $\mathrm{h}\mathcal{C}$ is a groupoid, we know that there exists a morphism $g : Y \to X$ of $\mathcal{C}$ such that $g \circ f$ and $\mathrm{Id}_X$ belong to the same path component of $\mathcal{C}(X, X)$, and $f \circ g$ and $\mathrm{Id}_Y$ belong to the same path component of $\mathcal{C}(Y, Y)$.

## 2.2 Simplicial categories

Now that we have the notion of $\infty$-groupoids defined, we need to investigate how it relates to topological spaces. As said before, we want to define a zigzag of Quillen equivalences from topological categories to topological spaces. This zigzag will go through the categories of simplicial sets. In this section, we will study simplicial categories, as an intermediate category which connects simplicial sets with topological categories.

**Definition 2.2.1.** A *simplicial category* is a category enriched over **sSet**. We will denote the category of all simplicial categories as **sSet-Cat**.

**Remark 2.2.2.** This definition of simplicial category is sometimes confused with the more general notion of a simplicial object in the category of categories. The category of all simplicial objects in the category of categories is denoted by **sCat**. Define the *simplicial inclusion* $\mathrm{I} : \mathbf{sSet\text{-}Cat} \to \mathbf{sCat}$ which sends any simplicial category $\mathcal{C}$ to a simplicial object in **Cat** with

$$\mathrm{Obj}(\mathrm{I}_n(\mathcal{C})) = \mathrm{Obj}(\mathcal{C}) \quad \text{and} \quad (\mathrm{I}_n(\mathcal{C}))(X,Y) = (\mathcal{C}(X,Y))_n \ \forall X,Y \in \mathrm{Obj}(\mathcal{C}).$$

Furthermore, there is an equivalence of categories between **sSet-Cat** and the full subcategory of **sCat** with the same objects at all levels.

The model structure over **sSet-Cat** can be defined by a "geometric realization", similarly to the construction of the model structure of simplicial sets. The following definition is a well-defined enriched functor thanks to the definition of **Top** as the category of compactly generated topological spaces, because we need the property proved in Proposition 1.4.16:

**Definition 2.2.3.** Define the *enriched geometric realization* $|\cdot|_e : \mathbf{sSet\text{-}Cat} \to \mathbf{Top\text{-}Cat}$ as the functor that sends every simplicial category $\mathcal{C}$ to the topological category $|\mathcal{C}|_e$ defined by:

- The same objects as $\mathcal{C}$.
- For every $X, Y \in \mathcal{C}$, $|\mathcal{C}|_e(X,Y) = |\mathcal{C}(X,Y)|$.
- The composition and identity are induced from the ones from $\mathcal{C}$ by applying the geometric realization functor.

**Definition 2.2.4.** Define the *homotopy category* $\mathrm{h}\mathcal{C}$ of a simplicial category $\mathcal{C}$ as $\mathrm{h}|\mathcal{C}|_e$.

**Definition 2.2.5.** A functor $F : \mathcal{C} \to \mathcal{D}$ between simplicial categories is a *Dwyer-Kan equivalence* if $|F| : |\mathcal{C}| \to |D|$ is a weak equivalence of topological categories.

**Remark 2.2.6.** This definition can be expressed in similar terms as the weak equivalences of topological categories. Thus, a functor $F : \mathcal{C} \to \mathcal{D}$ between simplicial categories is a Dwyer-Kan equivalence if and only if:

- For any $X, Y \in \mathcal{C}$, the map $\mathcal{C}(X,Y) \longrightarrow \mathcal{D}(FX,FY)$ is a weak equivalence in **sSet**.
- The induced morphism $\mathrm{h}F : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is an equivalence of categories.

For the definition of the cofibrations and cofibrant objects we need to define the generalization of free categories to the simplicial setting. This concept will be useful again later when we study simplicial localizations.

**Definition 2.2.7.** A category $\mathcal{C}$ is *free* if there is a set $S$ of non-identity maps in $\mathcal{C}$ such that every non-identity map in $\mathcal{C}$ can be uniquely written as a finite composition of maps in $S$. The maps of $S$ are called *generators*. Furthermore, a simplicial category $\mathcal{C}$ is *free* if it is a free category in each dimension after applying the simplicial inclusion $\mathrm{I}(\mathcal{C})$ and all degeneracies of generators are generators.

**Theorem 2.2.8.** [Ber08, Theorem 3.9 and Proposition 3.8]. *The category* **sSet-Cat** *admits a model structure called Bergner model structure such that:*

- *The weak equivalences are the Dwyer-Kan equivalences.*
- *The fibrations are the functors $F : \mathcal{C} \to \mathcal{D}$ such that $|F| : |\mathcal{C}| \to |\mathcal{D}|$ is a fibration of topological categories.*
- *The cofibrations are the morphism with the LLP with respect to any trivial fibration.*

*The fibrant objects are the categories in which the mapping simplicial sets are Kan complexes. On the other hand, the cofibrant objects are the retracts of a free simplicial category.*

Finally, we can define a homotopical inverse to the enriched geometric realization. Define the *enriched singular simplicial set* $\mathrm{Sing}_e : \mathbf{Top\text{-}Cat} \to \mathbf{sSet\text{-}Cat}$ as the functor which for any $\mathcal{C} \in \mathbf{Top\text{-}Cat}$, $\mathrm{Sing}_e(\mathcal{C})$ is a simplicial category with the same objects as $\mathcal{C}$ and that for each pair $X, Y \in \mathrm{Sing}_e(\mathcal{C})$, $(\mathrm{Sing}_e(\mathcal{C}))(X, Y) = \mathrm{Sing}(\mathcal{C}(X, Y))$. Using the Quillen equivalence defined in Theorem 1.4.20, it follows that those two functors induce a Quillen equivalence between topological and simplicial categories:

**Theorem 2.2.9.** [Amr13, Corollary 2.7]. *There is an adjunction*

$$|\cdot|_e : \mathbf{sSet\text{-}Cat} \rightleftarrows \mathbf{Top\text{-}Cat} : \mathrm{Sing}_e$$

*which is a Quillen equivalence.*

Then, we have proven a way to relate any topological category with a simplicial category respecting the homotopical structure. This is the first step of a series of Quillen equivalences that will end up relating the topological categories with topological spaces.

### 2.2.1 Homotopy coherent nerve

The next step is relating simplicial categories with simplicial sets. In this subsection we will define the pair of adjoint functors used to compare these two categories, using the general pattern of nerve and realization with a cosimplicial object.

**Definition 2.2.10.** Let $[n] \in \Delta$. Define the *homotopy coherent simplicial category* $\Delta^{\mathfrak{R}}[n] \in$ **sSet-Cat** as the simplicial category with:

- $\mathrm{Obj}(\Delta^{\mathfrak{R}}[n]) = [n] = \{0, \ldots, n\}$.
- For every pair of objects $i, j \in \mathrm{Obj}(\Delta^{\mathfrak{R}}[n])$:

$$\mathrm{Hom}(i, j) = \begin{cases} \emptyset & \text{if } i > j \\ N(P_{i,j}) & \text{if } i \leq j \end{cases}$$

  where $P_{i,j} = \{I \subseteq \mathcal{P}([n]) \mid \{i, j\} \subseteq I \subseteq \{i, i+1, \ldots, j\}\}$ as a poset category with set inclusion morphisms.

- For $i \leq j \leq k$, the composition is a map

$$\circ : \mathrm{Hom}(i,j) \times \mathrm{Hom}(j,k) \to \mathrm{Hom}(i,k)$$

induced by the map of posets:

$$\begin{array}{ccc} P_{i,j} \times P_{j,k} & \to & P_{i,k} \\ (I,J) & \mapsto & I \cup J \end{array}$$

**Remark 2.2.11.** (i) If $i < j$, it can be shown that $\mathrm{Hom}(i,j) = N(P_{i,j}) = (\Delta[1])^{(j-i-1)}$. When $j = i+1$, $\mathrm{Hom}(i,i)$ is the simplicial set with only one non-degenerate 0-simplex $\{i,j\}$. On the other hand, if $i = j$, $\mathrm{Hom}(i,i)$ is the simplicial set with only one non-degenerate 0-simplex $\{i\}$.

(ii) If $i \leq j$, $\mathrm{Hom}(i,j) = N(P_{i,j})$ has as 0-simplices the elements of $P_{i,j}$. Because the composition is given by union of sets at the level of 0-simplices, it is easy to see that all elements of $P_{i,j}$ except $\{i,j\}$ are decomposable. Then, each $\mathrm{Hom}(i,j)$ has one indecomposable 0-simplex, the one corresponding to the set $\{i,j\}$.

**Proposition 2.2.12.** *The map $[n] \mapsto \Delta^{\mathfrak{R}}[n]$ defines a covariant functor, i.e., $\Delta^{\mathfrak{R}}$ is a cosimplicial object in* **sSet-Cat**.

*Proof.* Given a non-decreasing morphism $f : [m] \to [n]$, we only need to define the induced morphism $f^* : \Delta^{\mathfrak{R}}[m] \to \Delta^{\mathfrak{R}}[n]$. At the level of the objects $f^*$ is simply the application of $f$. We just need to define a map $f^* : N(P_{i,j}) \to N(P_{f(i),f(j)})$ for every $i,j \in \mathbb{N}$. A map of nerves of posets is uniquely determined by the restriction to the 0-simplices. Then, we need to define a function $f^* : P_{i,j} \to P_{f(i),f(j)}$, and it is enough to define it only for the indecomposable elements. Thus, the desired map is defined by $f^*(\{i,j\}) = \{f(i), f(j)\}$. □

Thus, we can define a nerve and realization from the cosimplicial objects $\Delta^{\mathfrak{R}}[n]$ following the construction of Definitions 1.4.4 and 1.4.5. These functors are also known as:

**Definition 2.2.13.** The *homotopy coherent nerve* $\mathrm{N}^{\mathfrak{R}} :$ **sSet-Cat** $\to$ **sSet** is the functor which, for any $\mathcal{C} \in$ **sSet-Cat**,

$$\mathrm{N}^{\mathfrak{R}}_n(\mathcal{C}) := \mathrm{N}^{\Delta^{\mathfrak{R}}}_n = \mathbf{sSet\text{-}Cat}(\Delta^{\mathfrak{R}}[n], \mathcal{C}).$$

On the other hand, we define the *simplicial path category* $\mathfrak{C} :$ **sSet** $\to$ **sSet-Cat** as the functor which, for any $X \in$ **sSet**,

$$\mathfrak{C}(X) := |X|_{\Delta^{\mathfrak{R}}} = \int^{[n]\in\Delta} X_n \otimes \Delta^{\mathfrak{R}}[n].$$

By Proposition 1.4.6, $\mathrm{N}^{\mathfrak{R}}$ is right adjoint to $\mathfrak{C}$. In the following sections we will prove that in fact it is Quillen adjoint, but this requires introducing a new model structure over simplicial sets. First, we will present some examples of the application of $\mathfrak{C}$, and some of them will be useful later. The development of those examples follows the arguments given in [Hin07, Section 4.1.5]:

**Example 2.2.14** (Simplicial sphere)**.** Let $\mathcal{S}^n = \Delta[n]/\partial\Delta[n]$ be the simplicial $n$-th dimensional sphere. We want to compute $\mathfrak{C}(\mathcal{S}^n)$, which by definition is equal to

$$\mathfrak{C}(\mathcal{S}^n) = \int^{[n]\in\Delta} \mathcal{S}^n_n \otimes \Delta^{\mathfrak{R}}[n]. \tag{2.1}$$

Clearly, the objects of $\mathfrak{C}(\mathcal{S}^n)$ are the same as the ones from $\mathcal{S}^n$; therefore $\mathfrak{C}(\mathcal{S}^n)$ has only one object $*$. The morphisms of $\mathfrak{C}(\mathcal{S}^n)$ consist of only one mapping simplicial set $\mathfrak{C}(\mathcal{S}^n)(*,*)$. Because the only non-degenerate $k$-simplex with $k > 0$ of $\Delta[n]/\partial\Delta[n]$ is its only $n$-simplex, the only non-trivial coproduct of the coend of Equation 2.1 is the $n$-th one, which is equal to

$$\mathcal{S}^n_n \otimes \Delta^{\Re}[n] = * \otimes \Delta^{\Re}[n] = \Delta^{\Re}[n].$$

Then, by definition of coend, the morphisms of $\mathfrak{C}(\mathcal{S}^n)(*,*)$ must be morphisms of the simplicial category $\Delta^{\Re}[n]$ quotient by some relations dependent on the face maps of $\mathcal{S}^n$.

Then, the mapping simplicial set $\mathfrak{C}(\mathcal{S}^n)(*,*)$ is the free simplicial monoid generated by the set $\{[\alpha] \mid \alpha \in \Delta^{\Re}[n]\}$ modulo the following relations:

- $[\alpha \circ \beta] = [\alpha] \circ [\beta]$ if $\alpha$ and $\beta$ are composable.
- $[\alpha] = \mathrm{Id}_*$ if $\alpha$ belongs to the image of $\partial^i : \Delta^{\Re}[n-1] \to \Delta^{\Re}[n]$ for some $0 \le i \le n$.

Observe that the first condition ensures the compatibility between compositions, and the second one imposes the compatibility with the face maps of $\mathcal{S}^n$.

Let us try to explicitly compute that quotient. Recall that for each $a, b \in \Delta^{\Re}[n]$ we know $\Delta^{\Re}[n](a,b) = (\Delta[1])^{b-a-1}$. Then, each morphism $f : a \to b$ is represented by a tuple $(f_a, f_{a+1}, \dots, f_{b-1}, f_b)$ with all $f_i \in \Delta[1]$. Observe that in this case $f$ belongs to the image of a face map $\partial^i$ if and only if $i \notin (a,b)$ or $i \in (a,b)$ and $f_i = 1$. Thus, any morphism $f \in \Delta^{\Re}[n](a,b)$ with $a \neq 0$ or $b \neq n$ belongs to a face map. Furthermore, for $f \in \Delta^{\Re}[n](0,n)$, $[f] \neq \mathrm{Id}_*$ if and only if all $f_i \neq 1$. Then, observe that this description coincides with the definition of $\mathcal{S}^1 \wedge \dots \wedge \mathcal{S}^1$ ($n-1$ times). Therefore, $\mathfrak{C}(\mathcal{S}^n)(*,*)$ is homotopy equivalent to an $(n-1)$-sphere.

**Example 2.2.15** (Simplicial disk)**.** Let $\mathcal{D}^{n+1} = \Delta[n+1]/\bigcup_{i>0} \partial^i(\Delta[n])$ be the simplicial $(n+1)$-st dimensional disk. The same arguments as in the case of the simplicial sphere apply. The simplicial category $\mathfrak{C}(\mathcal{D}^{n+1})$ has only one object $*$, and one mapping simplicial set $\mathfrak{C}(\mathcal{D}^{n+1})(*,*)$. Furthermore, $\mathfrak{C}(\mathcal{D}^{n+1})(*,*)$ is also a free simplicial monoid generated by the set $\{[\alpha] \mid \alpha \in \Delta^{\Re}[n+1]\}$, but this time modulo the following relations:

- $[\alpha] = \mathrm{Id}_*$ if $\alpha = \beta \circ \gamma$ and $\beta : i \to n+1$ with $i > 1$.
- $[\alpha] = \mathrm{Id}_*$ if $\alpha$ belongs to the image of $\partial^i : \Delta^{\Re}[n-1] \to \Delta^{\Re}[n]$ for some $0 \le i \le n$.

By arguments similar to the ones of the previous example, we can prove that $\mathfrak{C}(\mathcal{D}^{n+1})(*,*)$ is $\Delta[1] \wedge \mathcal{S}^1 \wedge \dots \wedge \mathcal{S}^1$ (with $\mathcal{S}^1$ repeated $n-1$ times), which is homotopy equivalent to the $n$-th disk.

### 2.2.2 Joyal model structure

As we said before, the adjunction induced by the homotopy coherent nerve is a Quillen adjunction, but only if we consider another new model structure over simplicial sets:

**Theorem 2.2.16.** [Lur09, Theorem 2.2.5.1]. *There is a model structure over* **sSet** *called Joyal model structure with:*

- *The weak equivalences, called categorical equivalences, are the maps $f : X \to Y$ of simplicial sets such that $\mathfrak{C}(f) : \mathfrak{C}(X) \to \mathfrak{C}(Y)$ is a Dwyer-Kan equivalence.*
- *The cofibrations are the injective maps.*
- *The fibrations are the morphisms with the RLP with respect to any trivial cofibration.*

*Furthermore, all objects are cofibrant and the fibrant objects are the quasi-categories.*

**Remark 2.2.17.** By [JT07, Proposition 1.15], the Quillen model structure on **sSet** is a left Bousfield localization [Hir09, Definition 3.3.1] of the Joyal model structure. Thus, every categorical equivalence is a weak homotopy equivalence. Furthermore, a map between Kan complexes is a weak homotopy equivalence if and only if it is a categorical equivalence.

To distinguish the two different model categories over simplicial sets, we will denote by $\mathbf{sSet}_Q$ the one with the Quillen model structure, and by $\mathbf{sSet}_J$ the one with Joyal's model structure. Then, the adjunction induced by the homotopy coherent nerve forms a well-known Quillen equivalence:

**Theorem 2.2.18.** [Lur09, Theorem 2.2.5.1]. *The adjunction* $\mathfrak{C} : \mathbf{sSet}_J \rightleftarrows \mathbf{sSet}\text{-}\mathbf{Cat} : \mathrm{N}^{\mathfrak{R}}$ *determine a Quillen equivalence between* $\mathbf{sSet}_J$ *and* $\mathbf{sSet}\text{-}\mathbf{Cat}$.

**Proposition 2.2.19.** *The homotopy coherent nerve preserves weak equivalences of fibrant simplicial categories.*

*Proof.* By Theorem 2.2.18 and Proposition 1.2.7, $\mathrm{N}^{\mathfrak{R}}$ sends weak equivalences between fibrant simplicial categories to categorical weak equivalences. Also, we know that all categorical weak equivalences between simplicial sets are weak equivalences. Then, $\mathrm{N}^{\mathfrak{R}}$ sends weak equivalences between fibrant simplicial categories to weak equivalences. □

**Proposition 2.2.20.** *Let* $\mathcal{C}$ *be a fibrant simplicial category. If* $\mathrm{h}\mathcal{C}$ *is a groupoid, then* $\mathrm{N}^{\mathfrak{R}}(\mathcal{C})$ *is a Kan complex.*

*Proof.* By Theorem 2.2.18 and Proposition 1.2.8, we know that $\mathrm{N}^{\mathfrak{R}}$ sends fibrant simplicial categories to quasi-categories. Because $\mathcal{C}$ is a weak simplicial groupoid, the homotopy category of $\mathcal{C}$ is a groupoid. By [Joy02, Corollary 1.4], a quasi-category is a Kan complex if and only if its homotopy category is a groupoid. Then we only need to prove that the homotopy category of $\mathrm{h}(\mathrm{N}^{\mathfrak{R}}(\mathcal{C})) := \mathrm{h}(\mathfrak{C}(\mathrm{N}^{\mathfrak{R}}(\mathcal{C})))$ is a groupoid. Using Theorem 2.2.18, we know that the counit $\mathfrak{C}(\mathrm{N}^{\mathfrak{R}}(\mathcal{C})) \to \mathcal{C}$ is a Dwyer-Kan weak equivalence. Finally, this implies that $\mathrm{h}(\mathfrak{C}(\mathrm{N}^{\mathfrak{R}}(\mathcal{C}))) \cong \mathrm{h}\mathcal{C}$, which is a groupoid by the previous arguments. □

**Corollary 2.2.21.** *Let* $\mathcal{C}$ *be an* $\infty$*-groupoid. The simplicial set* $\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{C}))$ *is a Kan complex.*

*Proof.* For any topological category $\mathcal{C}$, $\mathrm{Sing}_e(\mathcal{C})$ is a fibrant simplicial category, because all topological categories are fibrant and $\mathrm{Sing}_e$ is a right Quillen adjoint. On the other hand, because $\mathcal{C}$ is an $\infty$-groupoid, $\mathrm{h}(\mathrm{Sing}_e(\mathcal{C}))$ is a groupoid. Then, applying the previous proposition, we obtain that $\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{C}))$ is a Kan complex. □

In order to relate the Joyal and the Quillen model structures, we will define another adjunction. We can use again the same pattern of nerve and realization:

**Proposition 2.2.22.** [JT07, Section 1]. *Let* $[n] \in \Delta$. *There is a cosimplicial object* $k : \Delta \to \mathbf{sSet}$ *defined as the nerve of the groupoid freely generated by* $[n]$ *as a category.*

Thus, we can define a nerve and realization from the cosimplicial object $k$ following the construction of Definitions 1.4.4 and 1.4.5. These functors are also known as:

**Definition 2.2.23.** The functor $k^! : \mathbf{sSet}_Q \to \mathbf{sSet}_J$ is defined for any $X \in \mathbf{sSet}_Q$ as

$$(k^!(X))_n := \mathrm{N}^k_n(X) = \mathbf{sSet}(k[n], X).$$

On the other hand, we define $k_! : \mathbf{sSet}_J \to \mathbf{sSet}_Q$ as the functor which, for any $X \in \mathbf{sSet}_J$,

$$k_!(X) := |X|_k = \int^{[n]\in\Delta} X_n \otimes k[n].$$

By Proposition 1.4.6, we know that these functors form an adjunction. But by [JT07, Theorem 1.19], we know there is in fact a Quillen adjunction

$$k_! : \mathbf{sSet}_Q \rightleftarrows \mathbf{sSet}_J : k^!.$$

This Quillen adjunction does not induce a Quillen equivalence. Finally, there are a couple of properties that help us relate these functors to general simplicial sets and Kan complexes:

**Proposition 2.2.24.** [JT07, Proposition 1.16]. *There is a functor $J$ from quasi-categories to Kan complexes, defined as $J(X)$ being the largest sub-Kan complex of a quasi-category $X$.*

**Proposition 2.2.25.** [JT07, Proposition 1.20]. *The following are true:*

(i) *The natural map $k^!(X) \to J(X)$ is a trivial fibration for every quasi-category $X$.*

(ii) *The natural map $X \to k_!(X)$ is monic and a weak homotopy equivalence for every simplicial set $X$.*

## 2.3 Grothendieck's homotopy hypothesis

In this section, we want to prove the homotopy hypothesis for the model presented above. By the results from Chapter 1, we know that there is a Quillen equivalence between topological spaces and simplicial sets. Then, we only need to prove that there is another Quillen equivalence between the category of $\infty$-groupoids and the category of simplicial sets, imposing the following zigzag of Quillen equivalences between categories. If $\psi = k^! \circ \mathrm{N}^{\mathfrak{R}} \circ \mathrm{Sing}_e$ and $\theta = |\cdot|_e \circ \mathfrak{C} \circ k_!$, the zigzag of Quillen equivalences is:

$$\mathbf{Top} \underset{\mathrm{Sing}}{\overset{|\cdot|}{\leftrightarrows}} \mathbf{sSet}_Q \underset{\psi}{\overset{\theta}{\rightleftarrows}} \infty\text{-}\mathbf{Grpd}.$$

This section follows the results presented in the article [Amr11]. First, we need to justify that there exists a model structure in the subcategory of $\infty$-groupoids. Then, we will prove that the Quillen adjunction between simplicial sets with the Quillen model structure and topological categories induces a Quillen equivalence when restricted to $\infty$-groupoids.

**Lemma 2.3.1.** *For every simplicial set $X$, $\theta(X)$ is an $\infty$-groupoid.*

*Proof.* Because $\theta$ is a left adjoint, it preserves colimits. By [Amr13, p.17], $\theta(\Delta[n])$ is an $\infty$-groupoid. Then, for any simplicial set $X$, $\theta(X)$ is an $\infty$-groupoid. □

**Lemma 2.3.2.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a map of $\infty$-groupoids. Then $F$ is a weak equivalence of topological categories if and only if $\psi(F)$ is a weak equivalence in $\mathbf{sSet}_Q$.*

*Proof.* Suppose that $F$ is a Dwyer-Kan equivalence. Because $\psi$ is a right Quillen functor and all objects in **Top-Cat** are fibrant, we know that $\psi(F)$ is a weak equivalence in $\mathbf{sSet}_Q$.

Now assume that $\psi(F)$ is a weak equivalence in $\mathbf{sSet}_Q$. By Corollary 2.2.21, $\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{C}))$ and $\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{D}))$ are Kan complexes. In addition, because $k^!$ is a Quillen right adjoint,

$\psi(\mathcal{C})$ and $\psi(\mathcal{D})$ are also Kan complexes. There is the following commutative diagram of weak equivalences:

$$\begin{array}{ccc} \psi(\mathcal{C}) & \xrightarrow{\sim} & \psi(\mathcal{D}) \\ \downarrow\sim & & \downarrow\sim \\ J(\mathrm{N}^{\Re}(\mathrm{Sing}_e(\mathcal{C}))) & \xrightarrow{\sim} & J(\mathrm{N}^{\Re}(\mathrm{Sing}_e(\mathcal{D}))) \\ \downarrow\sim & & \downarrow\sim \\ \mathrm{N}^{\Re}(\mathrm{Sing}_e(\mathcal{C})) & \xrightarrow{\sim} & \mathrm{N}^{\Re}(\mathrm{Sing}_e(\mathcal{D})) \end{array}$$

Furthermore, the maps $\psi(\mathcal{C}) \to \mathrm{N}^{\Re}(\mathrm{Sing}_e(\mathcal{C}))$ and $\psi(\mathcal{D}) \to \mathrm{N}^{\Re}(\mathrm{Sing}_e(\mathcal{D}))$ are trivial fibrations in $\mathbf{sSet}_Q$. By Remark 2.2.17, $\mathbf{sSet}_J$ is a left Bousfield localization of $\mathbf{sSet}_Q$, which implies that $\mathrm{N}^{\Re}(\mathrm{Sing}_e(\mathcal{C})) \to \mathrm{N}^{\Re}(\mathrm{Sing}_e(\mathcal{D}))$ is an equivalence of quasi-categories. Then, $\mathrm{Sing}_e(\mathcal{C}) \to \mathrm{Sing}_e(\mathcal{D})$ is a Dwyer-Kan equivalence of simplicial categories, and consequently, $F : \mathcal{C} \to \mathcal{D}$ is a topologically enriched weak equivalence. □

Using the previous lemmas, we can prove that there exists the induced model structure from the adjunction between simplicial sets and $\infty$-groupoids, and consider the homotopy hypothesis as a theorem:

**Theorem 2.3.3.** *The adjunction* $\theta : \mathbf{sSet}_Q \rightleftarrows \infty\text{-}\mathbf{Grpd} : \psi$ *induces a model structure on* $\infty$-**Grpd** *where:*

- *A morphism* $F : \mathcal{C} \to \mathcal{D}$ *of* $\infty$*-groupoids is a weak equivalence (fibration) if*

$$\psi(F) : \psi(\mathcal{C}) \to \psi(\mathcal{D})$$

  *is a weak equivalence (fibration) in* $\mathbf{sSet}_Q$.
- *The cofibrations are the morphisms with the LLP with respect to any trivial fibration.*

*Proof.* It is well-known that $\mathbf{sSet}_Q$ is cofibrantly generated (see [Hir09, Definition 11.1.2]) with generating cofibrations $I$ and generating trivial cofibrations $J$ being

$$I = \{\partial\Delta[n] \to \Delta[n]\}, \quad \text{and} \quad I = \{\Lambda^k[n] \to \Delta[n]\}.$$

By [Hir09, Theorem 11.3.2], the adjunction of $\theta$ and $\psi$ generates a cofibrantly generated right-induced model category on $\infty$-**Grpd** if:

(a) $\infty$-**Grpd** is complete and cocomplete.

(b) Both of the sets $\theta(I)$ and $\theta(J)$ permit the small object argument [Hir09, Definition 10.5.15].

(c) $\psi$ commutes with directed colimits.

(d) A transfinite composition of weak equivalences in $\mathbf{sSet}_Q$ is a weak equivalence.

(e) The pushout of a morphism from $\theta(J)$ by any morphism $f$ in $\infty$-**Grpd** is a weak equivalence.

First, consider the condition $(a)$. By [Amr13, Lemma 7.8.], we know that the inclusion of $\infty$-groupoids in topological categories has a right adjoint. Then, **Top-Cat** being complete implies that $\infty$-**Grpd** is also complete. Furthermore, because the functor $h$ : **Top-Cat** $\to$ **Cat** has a right adjoint (the inclusion), it commutes with colimits. Then, because the category of groupoids **Grps** is cocomplete, the restriction to $\infty$-groupoids $h : \infty\text{-}\mathbf{Grpd} \to \mathbf{Grps}$ proves that $\infty$-**Grpd** is cocomplete.

The condition $(b)$ follows directly from [Amr13, Lemma 2.5], and the condition $(d)$ is a well-known result. On the other hand, the condition $(c)$ follows from $k^!$ and $\mathrm{N}^{\mathfrak{R}}$ preserving colimits. Consider any pushout in $\infty$-**Grpd**:

$$\begin{array}{ccc} \theta(\Lambda^k[n]) & \longrightarrow & \mathcal{C} \\ \big\downarrow{\scriptstyle\sim} & \lrcorner & \big\downarrow{\scriptstyle f} \\ \theta(\Delta[n]) & \longrightarrow & \mathcal{D} \end{array}$$

By [Amr13, Lemma 2.2], $f$ is a weak equivalence of topological categories. Then, $\psi f$ is a weak equivalence in $\mathbf{sSet}_Q$ because of Lemma 2.3.2. □

**Theorem 2.3.4** (Grothendieck homotopy hypothesis)**.** *The Quillen adjunction*

$$\theta : \mathbf{sSet}_Q \rightleftarrows \infty\text{-}\mathbf{Grpd} : \psi$$

*is a Quillen equivalence.*

*Proof.* By [Hov07, Proposition 1.3.13], it is enough to prove that there is an equivalence of homotopical categories between $\mathbf{Ho}(\infty\text{-}\mathbf{Grpd})$ and $\mathbf{Ho}(\mathbf{sSet}_Q)$. First, we will prove that the functor $\mathrm{N}^{\mathfrak{R}} \circ \mathrm{Sing}_e : \infty\text{-}\mathbf{Grpd} \to \mathbf{sSet}_Q$ is well-defined. By Corollary 2.2.21, for every $\infty$-groupoid $\mathcal{C}$, $\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{C}))$ is a Kan complex. Because of Remark 2.2.17, the functor $\mathrm{N}^{\mathfrak{R}} \circ \mathrm{Sing}_e$ sends topological enriched weak equivalences (fibrations) to weak equivalences (fibrations) in $\mathbf{sSet}_Q$. Therefore, $\mathrm{N}^{\mathfrak{R}} \circ \mathrm{Sing}_e$ is a well-defined right Quillen functor.

Next, we want to prove that this right Quillen function induces an equivalence at the homotopy category. As it is well-known, it is enough to prove that the functors are fully faithful and essentially surjective. First, by the Quillen equivalence between **Top-Cat** and $\mathbf{sSet}_J$, there is an isomorphism $\mathbf{Top\text{-}Cat}(\mathcal{C}, \mathcal{D}) \to \mathbf{sSet}_J(\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{C})), \mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{D})))$. Also, because $\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{C}))$ and $\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{C}))$ are Kan complexes, we have an equality

$$\mathbf{sSet}_J(\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{C})), \mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{D}))) = \mathbf{sSet}_Q(\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{C})), \mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{D}))).$$

In addition, there is another equality $\mathbf{Top\text{-}Cat}(\mathcal{C}, \mathcal{D}) = \infty\text{-}\mathbf{Grpd}(\mathcal{C}, \mathcal{D})$, thanks to $\mathcal{C}$ and $\mathcal{D}$ being infinity groupoids and Lemma 2.3.2. Using these three facts, we know that there is an isomorphism $\infty\text{-}\mathbf{Grpd}(\mathcal{C}, \mathcal{D}) \to \mathbf{sSet}_Q(\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{C})), \mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{D})))$, which implies that $\mathrm{N}^{\mathfrak{R}} \circ \mathrm{Sing}_e$ is fully faithful.

Now we want to prove that $\mathrm{N}^{\mathfrak{R}} \circ \mathrm{Sing}_e$ is essentially surjective. From Proposition 2.2.25, we know that there is a natural transformation $X \to k_!(X)$ which is a weak equivalence of $\mathbf{sSet}_Q$. Then, the map

$$X \to k_!(x) \to \mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(|\,\mathfrak{C}(k_!(X))|_e))$$

is a weak equivalence of $\mathbf{sSet}_Q$, thanks to the unit map from the Quillen equivalence between **Top-Cat** and $\mathbf{sSet}_J$. Finally, because $|\,\mathfrak{C}(k_!(X))|_e$ is an infinity groupoid, we know that $\mathrm{N}^{\mathfrak{R}} \circ \mathrm{Sing}_e$ is essentially surjective.

On the other hand, for any infinity groupoid $\mathcal{C}$, using Proposition 2.2.25, we know that there is a trivial fibration $k^!(\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{C}))) \to J(\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{C}))) = \mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathcal{C}))$. Then, the functor

$$\psi = k^! \circ \mathrm{N}^{\mathfrak{R}} \circ \mathrm{Sing}_e : \mathbf{Ho}(\infty\text{-}\mathbf{Grpd}) \to \mathbf{Ho}(\mathbf{sSet}_Q)$$

is an equivalence of homotopical categories, with left adjoint $\theta = |\cdot|_e \circ \mathfrak{C} \circ k_!$. Therefore, the adjunction $\theta : \mathbf{sSet}_Q \rightleftarrows \infty\text{-}\mathbf{Grpd} : \psi$ induces a Quillen equivalence. □

The previous theorem proves that $\infty$-groupoids, as defined in Section 2.1, are homotopy equivalent to topological spaces. In particular, it also proves that for every topological space $X$ there exists an $\infty$-groupoid defined by:

$$\Pi_\infty(X) := \theta(\mathrm{Sing}(X)) = |\,\mathfrak{C}(k_!(\mathrm{Sing}(X)))|_e.$$

By the previous theorem and Theorem 1.4.20, we know that $|\psi(\Pi_\infty(X))| \simeq X$, which means that $\Pi_\infty(X)$ has the same homotopy type as the fundamental $\infty$-groupoid of $X$.

## 2.4 Moore path category

In the previous section, we have shown that there is a model for the homotopy type of a topological space $X$ as an $\infty$-groupoid. But this construction is not explicit. It is defined as the composition of non-trivial functors, like $\mathfrak{C}$. In this section, we want to show that there exist an explicit construction, which is defined as a topological category directly using the information of the associated topological space.

Recall that a pointed space is a tuple $(X, x)$ with $x \in X$. Furthermore, denote $\mathbb{R}_+ := [0, \infty)$, which can be viewed as a topological space with the Euclidean topology.

**Definition 2.4.1.** Let $X$ be a topological space and $x, y \in X$. Define the *Moore path space* in $X$ between $x$ and $y$ as

$$P^M_{x,y}X = \{(f, r) \in X^{\mathbb{R}_+} \times \mathbb{R}_+ \mid f(0) = x \text{ and } f(s) = y \ \forall s \geq r\},$$

with the product topology induced by the Euclidean topology on $\mathbb{R}_+$ and the compact-open topology on $X^{\mathbb{R}_+}$. For any element $(f, r) \in P^M_{x,y}X$, $r$ will be called the *length* of the path $f$. Furthermore, the *Moore loop space* of a pointed space $(X, x)$ is defined as $\Omega^M_x X = P^M_{x,x}X$.

The standard path space $P_{x,y}X$ can be embedded into the Moore path space, assigning always a length of 1 to each path. From this fact, the following properties follow:

**Proposition 2.4.2.** *Let $X$ be a topological space and $x, y \in X$. Then:*

(i) *The path space $P_{x,y}X$ is a deformation retract of the Moore path space $P^M_{x,y}X$. In particular, $P_{x,y}X$ and $P^M_{x,y}X$ are homotopy equivalent.*

(ii) *The loop space $\Omega_x X$ is a deformation retract of the Moore loop space $\Omega^M_x X$. In particular, $\Omega_x X$ and $\Omega^M_x X$ are homotopy equivalent.*

*Proof.* Observe that $P_{x,y}X \subset P^M_{x,y}X$. Then, there is a deformation retract defined by the inclusion $i : P_{x,y}X \to P^M_{x,y}X$ and the retraction $r : P^M_{x,y}X \to P_{x,y}X$ with $r(f, r)(t) = f(t/r)$. We clearly have $r \circ i = \mathrm{Id}_{P_{x,y}X}$ and $i \circ r \sim \mathrm{Id}_{P^M_{x,y}X}$, which implies that $P_{x,y}X$ is a deformation retract of $P^M_{x,y}X$. Then, $P_{x,y}X$ and $P^M_{x,y}X$ must be homotopy equivalent. The same argument works with $\Omega_x X$ and $\Omega^M_x X$. □

Define a *topological monoid* as a monoid object in the category of topological spaces. Let **tMon** be the category of all topological monoids. A topological monoid $G$ is *group-like* if the set of path connected components $\pi_0(G)$ is a group with respect to the multiplication induced by the monoid structure of $G$.

Then, consider the following composition operation:

$$\begin{aligned} \circ : \; P^M_{x,y}X \times P^M_{y,z}X &\longrightarrow P^M_{x,z}X \\ ((f,r),(g,s)) &\longmapsto (f * g, r+s) \end{aligned} \tag{2.2}$$

where $f * g$ denotes the path composition defined by

$$(f*g)(t) = \begin{cases} f(t) & \text{if } 0 \le t < r \\ g(t-r) & \text{if } t \ge r \end{cases}$$

This operation induces a structure over the Moore path spaces, which is stricter than the one from the standard path spaces:

**Proposition 2.4.3.** *The operation of Equation 2.2 is strictly associative, strictly unitary and weakly invertible. In particular, for any topological space $X$, $\Omega^M_x X$ is a group-like topological monoid with this operation.*

*Proof.* The strict associativity and unity follow directly from the definition. To check the weak invertibility, consider an element $(f,r) \in P^M_{x,y}X$. Then define an element $(g,r) \in P^M_{y,x}X$ by

$$g(t) = \begin{cases} f(r-t) & \text{if } 0 \le t \le r \\ x & \text{if } t > r \end{cases}$$

Thus, $(f*g, 2r)$ belongs to the same path component of $\Omega^M_y X$ as $(c_y, 0)$. The same happens with $(g*f, 2r)$ and $(c_x, 0)$. □

Observe that the previous property induces a monoidal structure on path spaces. This structure can be realized as a topological category:

**Definition 2.4.4.** Let $X$ be a topological space. Define the *Moore path category* $\Pi^M_\infty(X)$ as the topological category with:

- The objects of $\Pi^M_\infty(X)$ are the points of $X$.
- For any two objects $x, y \in \Pi^M_\infty(X)$, the morphisms are $\Pi^M_\infty(X)(x,y) = P^M_{x,y}X$.
- For any two objects $x, y, z \in \Pi^M_\infty(X)$, the composition is given by Equation 2.2.
- For every object $x \in \Pi^M_\infty(X)$, the identity is the constant path $(c_x, 0)$.

**Remark 2.4.5.** The Moore path category construction is in fact a functor. Any continuous function $f : X \to Y$ induces a functor between topological categories $\Pi^M_\infty(f) : \Pi^M_\infty(X) \to \Pi^M_\infty(Y)$ defined on objects by applying $f$, and on each homset by:

$$\begin{aligned} \Pi^M_\infty(X)(x,y) &\longrightarrow \Pi^M_\infty(Y)(f(x), f(y)) \\ (g,r) &\longmapsto (f \circ g, r) \end{aligned}$$

**Remark 2.4.6.** For any point $x \in X$, we can consider the topological category $\Pi^M_\infty(\{x\})$, which has one object $x$ and only one homset $\Pi^M_\infty(\{x\})(x,x) = \Omega^M_x X$. On the other hand, define the *delooping* $\mathbb{D} : \mathbf{tMon} \to \mathbf{Top\text{-}Cat}_0$ as the functor that sends a topological monoid $M$ to the topological category with one object $*$ and $\mathrm{Hom}(*,*) = M$. Then, observe that we can define trivial functors from $\mathbb{D}(\Omega^M_x X)$ to $\Pi^M_\infty(\{x\})$, which send the object $*$ to $x$, and act as the identity on the homset. In fact, these functors induce an equivalence of categories, i.e., $\mathbb{D}(\Omega^M_x X) \cong \Pi^M_\infty(\{x\})$.

Because we expect $\Pi_\infty^M(X)$ to be a model for the fundamental $\infty$-groupoid of the space $X$, in particular, it should be an $\infty$-groupoid:

**Proposition 2.4.7.** *For every topological space $X$, $\Pi_\infty^M(X)$ is an $\infty$-groupoid.*

*Proof.* This is a direct consequence of the weak invertibility of the Moore paths, proved in Proposition 2.4.3, and the definition of the homotopy category. □

Before continuing our constructions, we need a new definition. A pointed space $(X, x)$ is called *well-pointed* if the inclusion $\{x\} \hookrightarrow X$ is a Hurewicz cofibration. In particular, any pointed CW-complex is a well-pointed space. Furthermore, a topological monoid $M$ with identity element $e$ is *well-pointed* if the pointed topological space $(M, e)$ is well-pointed.

The goal of the rest of this section is to prove that the $\infty$-groupoid $\Pi_\infty^M(X)$ has the homotopy type of the fundamental $\infty$-groupoid of the topological space $X$. Our proof is inspired by [McG20], where the author uses the argument of [RZ18, Proposition 7.2]. This argument includes the following claim:

**Main Theorem.** *Let $(X, x)$ be a path-connected well-pointed topological space. The topological space $|\,\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathbb{D}\,\Omega_x^M(X)))|$ is a classifying space for $\Omega_x^M(X)$, and as a consequence, there is a natural weak homotopy equivalence*

$$|\,\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathbb{D}\,\Omega_x^M X))| \simeq X.$$

We have not been able to find any reference proving this fact. As explained in the Introduction, Chapter 3 will be devoted to introducing the theory of classifying spaces and proving this theorem in detail. Thus, we can assume this theorem as proven, and complete the last proof of this section:

**Theorem 2.4.8.** *The $\infty$-groupoid $\Pi_\infty^M(X)$ is a model for the homotopy type of the topological space $X$.*

*Proof.* By definition of $\Pi_\infty^M$, the connected components of $X$ induce disconnected topological subcategories in $\Pi_\infty^M(X)$. Therefore, it is enough to consider $X$ being connected. Then, the desired weak equivalence will be the disjoint union of the weak equivalences between each connected component.

Thus, we assume that $X$ is connected. By Remark 2.4.6, for any point $x \in X$, the $\infty$-groupoid $\Pi_\infty^M(\{x\})$ is equivalent to $\mathbb{D}\,\Omega_x^M(X)$. Also, it is a subcategory of $\Pi_\infty^M(X)$. The inclusion functor $\Pi_\infty^M(\{x\}) \hookrightarrow \Pi_\infty^M(X)$ is fully faithful because it induces the identity of the homset of $x$. Furthermore, it is essentially surjective at the homotopy categories, which implies that the inclusion is in fact a weak equivalence. To sum up, studying $\Pi_\infty^M(X)$ up to weak equivalence when $X$ is connected is equivalent to studying $\Pi_\infty^M(\{x\}) \cong \mathbb{D}\,\Omega_x^M(X)$.

Now we want to use the previously mentioned theorem. First observe that the well-pointed hypothesis can be assumed, because if $X$ is not well-pointed, $|\,\mathrm{Sing}(X)|$ will be well-pointed and weakly equivalent to $X$. By the Main Theorem, because $X$ is connected, we know that

$$|\,\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathbb{D}\,\Omega_x^M X))| \simeq X.$$

Then, applying Sing to the weak equivalence and composing with the unit map, we obtain

$$\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathbb{D}\,\Omega_x^M X)) \simeq \mathrm{Sing}(X).$$

Because $\mathrm{N}^{\mathfrak{R}} \circ \mathrm{Sing}_e$ is a right Quillen functor, it preserves weak equivalences between fibrant objects. But all topological categories are fibrant, therefore, it preserves all weak equivalences. Then, using $\Pi_\infty^M(X) \simeq \Pi_\infty^M(\{x\}) \cong \mathbb{D}\,\Omega_x^M(X)$, we have

$$\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\Pi_\infty^M(X))) \simeq \mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathbb{D}\,\Omega_x^M(X))).$$

By Proposition 2.2.25, for any Kan complex $K$, we have $k^!(K) \xrightarrow{\sim} J(K) = K$. Then, using Theorem 2.3.4,

$$k^!(\mathrm{N}^{\Re}(\mathrm{Sing}_e(\mathbb{D}\,\Omega_x^M(X)))) \simeq \mathrm{N}^{\Re}(\mathrm{Sing}_e(\mathbb{D}\,\Omega_x^M(X))) \simeq \mathrm{Sing}(X).$$

Finally, recall that all simplicial sets are cofibrant, and since $|\cdot|$ is a left adjoint, it preserves all weak equivalences. Thus, using the counit map,

$$|k^!(\mathrm{N}^{\Re}(\mathrm{Sing}_e(\Pi_\infty^M(X))))| \simeq |k^!(\mathrm{N}^{\Re}(\mathrm{Sing}_e(\mathbb{D}\,\Omega_x^M(X))))| \simeq X.$$

Finally, because $|k^!(\mathrm{N}^{\Re}(\mathrm{Sing}_e(\cdot)))|$ is a right Quillen functor, $\Pi_\infty^M(X)$ is a model of the homotopy type of $X$. In particular, we have proven that $\Pi_\infty(X) = |\,\mathfrak{C}(k_!(\mathrm{Sing}(X)))|_e$ and $\Pi_\infty^M(X)$ are weakly equivalent as ∞-groupoids.

□

# Chapter 3

# Models of the classifying space

As defined in the Introduction, a *classifying space* $\mathrm{B}(G)$ of a topological group $G$ is defined as the quotient of a weakly contractible space $\mathrm{E}(G)$ by a proper free action of $G$. The first functorial construction of classifying spaces was due to Milnor [Mil56]. Later, Dold and Lashof [DL59] presented a different functorial construction inspired by Milnor, which applied also to any topological group-like monoid. Finally, Milgram [Mil67] refined Dold and Lashof's work, by formalizing the nowadays well-known bar construction.

In the previous chapter, we have been able to reduce the correctness of the model of the fundamental $\infty$-groupoid as a Moore path category to the following theorem about classifying spaces:

**Main Theorem.** *Let $(X, x)$ be a path-connected well-pointed topological space. The topological space $|\,\mathrm{N}^{\Re}(\mathrm{Sing}_e(\mathbb{D}\,\Omega_x^M(X)))|$ is a classifying space for $\Omega_x^M(X)$, and as a consequence, there is a natural weak homotopy equivalence*

$$|\,\mathrm{N}^{\Re}(\mathrm{Sing}_e(\mathbb{D}\,\Omega_x^M X))| \simeq X.$$

The goal of this chapter is to prove this result in detail. Let $M$ be a topological group-like monoid. In the first section, we review the functorial classifying space $\mathrm{B}(M)$ as defined by Milgram [Mil67]. The section is written following the survey of May [May75] about the Milgram classifying space. The second section explores another functorial classifying space based on a simplicial nerve different from the homotopy coherent nerve, the diagonal simplicial nerve $\mathrm{N}^d : \mathbf{sSet\text{-}Cat} \to \mathbf{sSet}$. Using this nerve, the *diagonal nerve classifying space* is defined as the functor

$$M \mapsto |\,\mathrm{N}^d(\mathrm{Sing}_e(\mathbb{D}\,M))|.$$

The aim of the second section is to prove that the diagonal nerve classifying space has the same homotopy type as the Milgram classifying space, i.e., $|\,\mathrm{N}^d(\mathrm{Sing}_e(\mathbb{D}\,M))| \simeq \mathrm{B}(M)$. Finally, in the third section we define a functorial classifying space

$$M \mapsto |\,\mathrm{N}^{\Re}(\mathrm{Sing}_e(\mathbb{D}\,M))|.$$

The proof of the Main Theorem follows from the equivalence $|\,\mathrm{N}^{\Re}(\mathrm{Sing}_e(\mathbb{D}\,M))| \simeq \mathrm{B}(M)$. In order to prove this fact, we use the diagonal nerve classifying space and prove a weak equivalence $\mathrm{N}^{\Re}(\mathcal{C}) \simeq \mathrm{N}^d(\mathcal{C})$ for every fibrant simplicial category $\mathcal{C}$ with $\mathrm{h}\mathcal{C}$ a groupoid. Therefore, the main ideas behind the proof are shown in the following scheme:

$$\left.\begin{aligned} |\,\mathrm{N}^d(\mathrm{Sing}_e(\mathbb{D}\,M))| \simeq \mathrm{B}(M) \\ \mathrm{N}^{\Re}(\mathcal{C}) \simeq \mathrm{N}^d(\mathcal{C}) \end{aligned}\right\} \rightsquigarrow |\,\mathrm{N}^{\Re}(\mathrm{Sing}_e(\mathbb{D}\,M))| \simeq \mathrm{B}(M)$$

## 3.1 Classifying space of a monoid

First, we need to introduce the category of simplicial spaces **sTop** := Func($\Delta^{op}$, **Top**), i.e., the category of simplicial objects internal to topological spaces. Given any topological monoid, we can build a simplicial space with the following well-known construction:

**Definition 3.1.1.** Let $M$ be a topological monoid with identity element $e$, and $X$, $Y$ be two topological spaces such that $X$ is a right $M$-space, and $Y$ is a left $M$-space. The *bar construction* $\widetilde{\mathrm{B}}(X, M, Y)$ is a simplicial space with $X \times M^n \times Y$ as the space of $n$-simplices, and for every $0 \leq i \leq n$, the faces and degeneracies defined by

$$\partial_i(x, m_1, \ldots, m_n, y) = \begin{cases} (x \cdot m_1, m_2, \ldots, m_n, y) & \text{if } i = 0 \\ (x, m_1, \ldots, m_{i-1}, m_i \cdot m_{i+1}, m_{i+2}, \ldots, m_n, y) & \text{if } 1 \leq i < n \\ (x, m_1, \ldots, m_{n-1}, m_n \cdot y) & \text{if } i = n \end{cases}$$

$$s_i(x, m_1, \ldots, m_n, y) = (x, m_1, \ldots, m_i, e, m_{i+1}, \ldots, m_n, y)$$

Let $*$ denote the singleton topological space. Using the bar construction, we can define two simplicial spaces associated to any topological monoid $M$, $\widetilde{\mathrm{B}}(M) := \widetilde{\mathrm{B}}(*, M, *)$ and $\widetilde{\mathrm{E}}(M) := \widetilde{\mathrm{B}}(*, M, M)$. From this definition we can explicitly compute the simplices of those simplicial spaces:

$$\widetilde{\mathrm{B}}_0(M) = * \text{ and } \widetilde{\mathrm{B}}_n(M) = M^n, \qquad \widetilde{\mathrm{E}}_0(M) = M \text{ and } \widetilde{\mathrm{E}}_n(M) = M^{n+1}.$$

The next step is obtaining a topological space related to those simplicial spaces. Given any simplicial space, we can make a construction which is very similar to the geometric realization from simplicial sets to topological spaces:

**Definition 3.1.2.** For any simplicial space $X \in$ **sTop**, we define the *topological geometric realization* of $X$ as

$$|X|_t = \int^{[n]\in\Delta} X_n \times \Delta^n,$$

where the inner product is the one from **Top**.

Then, using the topological geometric realization, we can finally define the classifying space of any topological monoid:

**Definition 3.1.3.** The *Milgram classifying space* B : **tMon** → **Top** is the composition

$$\mathrm{B}(M) := |\,\widetilde{\mathrm{B}}(M)|_t = \int^{n\in\Delta} \widetilde{\mathrm{B}}_n(M) \times \Delta^n = \int^{n\in\Delta} M^n \times \Delta^n.$$

Let **Top-Cat**$_0$ ⊂ **Top-Cat** be the category of topological categories with only one object, **sTop**$_0$ ⊂ **sTop** the category of simplicial spaces with a singleton as the 0-th level space. There is another equivalent definition of this construction that goes back to Segal [Seg68]. First, consider the delooping functor $\mathbb{D}$ as defined in Remark 2.4.6. On the other hand, we can apply the well-known nerve construction of a category to topological categories. The *topological nerve* $\mathrm{N}^t$ : **Top-Cat**$_0$ → **sTop**$_0$ is a functor that for any $\mathbb{D}\,M \in$ **Top-Cat**$_0$ with $\mathrm{Hom}(*, *) = M$:

$$\mathrm{N}^t_0(\mathbb{D}\,M) = * \text{ and } \mathrm{N}^t_n(\mathbb{D}\,M) = M^n.$$

Thus, the previous definition of the bar construction for $\widetilde{\mathrm{B}}(M)$ coincides with the following composition:

$$\widetilde{\mathrm{B}}(M) = (\mathrm{N}^t \circ \mathbb{D})(M). \tag{3.1}$$

This alternative definition of the simplicial space $\widetilde{\mathrm{B}}(M)$ will be useful in the following sections.

In addition to the functorial construction, Milgram also proved that if the topological monoid is group-like, then there is a natural weak equivalence relating the homotopy type of $M$ and its classifying space:

**Theorem 3.1.4.** [Mil67]. *For any group-like topological monoid $M$, the natural map $M \mapsto \Omega\,\mathrm{B}(M)$ is a weak equivalence.*

**Example 3.1.5.** Consider the Moore loop space $\Omega_x^M X$, for some pointed topological space $(X, x)$. By Proposition 2.4.3, we know that $\Omega_x^M X$ is a group-like topological monoid. The classifying space of $\Omega_x^M X$ is

$$\mathrm{B}(\Omega_x^M X) = |\,\widetilde{\mathrm{B}}(\Omega_x^M X)|_t.$$

Therefore, by Theorem 3.1.4, we know that the natural map $\Omega_x^M X \mapsto \Omega\,\mathrm{B}(\Omega_x^M X)$ is a weak equivalence. Furthermore, this example has an exclusive property related to $\Omega_x^M X$ being a loop space:

**Proposition 3.1.6.** *Let $(X, x)$ be a path-connected pointed topological space. Then, there is a natural weak homotopy equivalence*

$$\mathrm{B}(\Omega_x^M X) \simeq X.$$

*Proof.* The proof is a consequence of the commutativity of the following diagram:

$$\begin{array}{ccccccc}
\Omega_x^M X & = & \Omega_x^M X & = & \Omega_x^M X & = & \Omega_x^M X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P_{x,\bullet}^M X & \longleftarrow & |\,\widetilde{\mathrm{B}}(P_{x,\bullet}^M X, \Omega_x^M X, \Omega_x^M X)|_t & \xrightarrow{p^*} & |\,\widetilde{\mathrm{B}}(*, \Omega_x^M X, \Omega_x^M X)|_t & = & \mathrm{E}(\Omega_x^M X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \xleftarrow{e^*} & |\,\widetilde{\mathrm{B}}(P_{x,\bullet}^M X, \Omega_x^M X, *)|_t & \xrightarrow{q^*} & |\,\widetilde{\mathrm{B}}(*, \Omega_x^M X, *)|_t & = & \mathrm{B}(\Omega_x^M X)
\end{array}$$

where $p^*$ and $q^*$ are induced from the weak equivalence $P_{x,\bullet}^M X \to *$, thanks to $P_{x,\bullet}^M X$ being weakly contractible; and $e^*$ is induced from the evaluation map $e : P_{x,\bullet}^M X \to X$ with $e(f, r) = f(r)$. Because $X$ is path connected, $e^*$ is also a weak equivalence, and the bottom row is a zigzag of weak equivalences. Finally, because $X$ is path connected, there is a map from $\mathrm{B}(\Omega_x^M X)$ to $\Omega_x^M X$, which gives rise to the target map from $\mathrm{B}(\Omega_x^M X)$ to $X$, which must be a weak equivalence by the two-out-of-three property. □

## 3.2 Diagonal simplicial nerve model

As previously mentioned, in this section we want to study a functorial classifying space called the *diagonal nerve classifying space*, which is homotopy equivalent to the Milgram classifying space. The name of this model comes from the main functor used in its definition: the diagonal simplicial nerve $\mathrm{N}^d : \mathbf{sSet\text{-}Cat} \to \mathbf{sSet}$. Unlike the homotopy coherent nerve, the diagonal simplicial nerve does not induce a Quillen equivalence between simplicial categories and simplicial sets. It will be defined similarly by the nerve and realization pattern, as a nerve arising from a suitable cosimplicial object.

But there is an alternative definition, which predates the existence of the homotopy coherent nerve, and helps to give an intuition of how the diagonal simplicial nerve works.

Take any simplicial category $\mathcal{C} \in \mathbf{sSet\text{-}Cat}$, and consider its simplicial inclusion $\mathrm{I}(\mathcal{C}) \in \mathbf{sCat}$ as defined in Remark 2.2.2. Then, apply the classical nerve of a category levelwise, obtaining a bisimplicial set. Finally, take the diagonal of this bisimplicial set and define the resulting simplicial set as the image $\mathrm{N}^d(\mathcal{C})$. This simplicial nerve appeared in the initial work of Dwyer and Kan [DK80] about simplicial categories.

To obtain the functorial classifying space from $\mathrm{N}^d$ we just need to compose it with other previously defined functors. First, precompose with $\mathrm{Sing}_e \circ \mathbb{D}$, which sends any topological monoid to a simplicial category. Then, apply the geometric realization to the image of $\mathrm{N}^d$, obtaining a topological space. Thus, we define the diagonal nerve classifying space as the functor $M \mapsto |\,\mathrm{N}^d(\mathrm{Sing}_e(\mathbb{D}(M)))|$ for any topological monoid $M$. The goal of this section is to prove that under some technical conditions on $M$ there is a homotopy equivalence

$$|\,\mathrm{N}^d(\mathrm{Sing}_e(\mathbb{D}(M)))| \simeq \mathrm{B}(M). \tag{3.2}$$

The proof of this fact will be proven in two subsections. In the first one, we study the relation between the topological geometric realization and the usual geometric realization. In the second one, we define the diagonal simplicial nerve, show its main properties, and prove Equation 3.2 using the results from the previous subsection.

### 3.2.1 Topological geometric realization

In this subsection we want to prove a weakly homotopy equivalent factorization of the topological geometric realization through the classical geometric realization. This weak homotopical factorization will be used later in the proof of the diagonal nerve classifying space.

For a general simplicial space, the topological geometric realization does not preserve levelwise weak equivalences as weak equivalences of topological spaces. This fact motivates the definition of an alternative geometric realization which will have the desired property of preserving levelwise weak equivalences:

**Definition 3.2.1.** For any simplicial space $X \in \mathbf{sTop}$, we define the *fat geometric realization* as

$$\|X\|_t = \int^{[n]\in\Delta^+} X_n \times \Delta^n$$

where $\Delta^+$ denotes the subcategory of $\Delta$ without the degeneracy maps (sometimes called Delta sets), and the product is the one from **Top**.

**Proposition 3.2.2.** [Seg74, Proposition A.1.(ii)]. *Let $X, Y \in \mathbf{sTop}$. If $f : X \to Y$ is a levelwise weak homotopy equivalence, then the induced map $\|X\|_t \to \|Y\|_t$ is a weak homotopy equivalence of topological spaces.*

Next, consider a subclass of all simplicial spaces, the *good* simplicial spaces, which can be proven to have weakly equivalent geometric realizations. In particular, good simplicial spaces preserve levelwise weak equivalence with any of the two geometric realizations.

**Definition 3.2.3.** A simplicial space $X$ is *good* if all degeneracy maps $d_n : X_{n-1} \to X_n$ are closed Hurewicz cofibrations.

**Proposition 3.2.4.** [Seg74, Proposition A.1.(iv)]. *If a simplicial space $X$ is good, then $\|X\|_t \simeq |X|_t$.*

**Proposition 3.2.5.** *Let $f : X \to Y$ be a levelwise weak homotopy equivalence of simplicial spaces. If $X$ and $Y$ are good simplicial spaces, then the induced map $|X|_t \to |Y|_t$ is a weak homotopy equivalence of topological spaces.*

*Proof.* Follows directly from Proposition 3.2.2 and Proposition 3.2.4. □

Because we are working with simplicial spaces obtained using the bar construction, it is natural to ask under which conditions on the topological monoid the bar construction is good.

**Proposition 3.2.6.** *If $M$ is a well-pointed monoid, then $\widetilde{\mathrm{B}}(M)$ is a good simplicial space.*

*Proof.* By [Tsu16, Lemma 5.5.(i)], we know that all degeneracy maps $d_n : \widetilde{\mathrm{B}}_{n-1}(M) \to \widetilde{\mathrm{B}}_n(M)$ are Hurewicz cofibrations. Because we are only considering compactly generated weakly Hausdorff spaces, any Hurewicz cofibration is closed, and therefore $\widetilde{\mathrm{B}}(M)$ is a good simplicial space. □

Let $(X, x)$ be a well-pointed topological space as defined in Section 2.4. Then, $\Omega_x^M X$ is a well-pointed topological monoid. Using the previous property, we know that $\widetilde{\mathrm{B}}(\Omega_x^M X)$ is a good simplicial space.

Now we introduce functors to translate between simplicial spaces and bisimplicial sets. Those functors are basically a levelwise application of the usual singular simplicial set and geometric realization:

**Definition 3.2.7.** The *levelwise singular chain* functor $\mathrm{Sing}_\ell : \mathbf{sTop} \to \mathbf{bSet}$ for all $X \in \mathbf{sTop}$ and all $n, m \in \mathbb{N}$ is

$$(\mathrm{Sing}_\ell(X))_{n,m} = \mathrm{Sing}_m(X_n).$$

Similarly, the *levelwise geometric realization* $|\cdot|_\ell : \mathbf{bSet} \to \mathbf{sTop}$ for all $S \in \mathbf{bSet}$ and all $n \in \mathbb{N}$ is

$$(|S|_\ell)_n = |S_{n,\bullet}| = \int^{n\in\Delta} S_{n,\bullet} \otimes \Delta^n.$$

**Lemma 3.2.8.** [Seg74, Proposition A.3]. *For every simplicial space $X$, $|\mathrm{Sing}_\ell(X)|_\ell$ is good.*

**Lemma 3.2.9.** *Let $X \in \mathbf{sTop}$ be a simplicial space. The natural morphism*

$$|\mathrm{Sing}_\ell(X)|_\ell \to X$$

*is a levelwise weak homotopy equivalence of topological spaces.*

*Proof.* By Proposition 1.4.19, the counit map $\epsilon_{X_n} : |\mathrm{Sing}_\ell(X_n)|_\ell \to X_n$ is a weak equivalence of topological spaces for every $n \in \mathbb{N}$. □

On the other hand, we can consider two different functors between bisimplicial sets and simplicial sets. After the definitions, we give two very well-known properties of these functors:

**Definition 3.2.10.** Define the following functors:

(i) The *diagonal* $\mathrm{d} : \mathbf{bSet} \to \mathbf{sSet}$ for all $S \in \mathbf{bSet}$ and all $n \in \mathbb{N}$ as $(\mathrm{d}(S))_n = S_{n,n}$.

(ii) The *simplicial geometric realization* $|\cdot|_s : \mathbf{bSet} \to \mathbf{sSet}$ for all $S \in \mathbf{bSet}$ and all $n \in \mathbb{N}$ as

$$(|S|_s)_n = \int^{n\in\Delta} S_{n,\bullet} \times \Delta[n].$$

**Proposition 3.2.11.** [GJ09, Proposition 1.7]. *If a map $f : S \to R$ of bisimplicial spaces is a levelwise weak equivalence, then the induced map $f_* : \mathrm{d}(S) \to \mathrm{d}(R)$ is a weak equivalence of simplicial sets.*

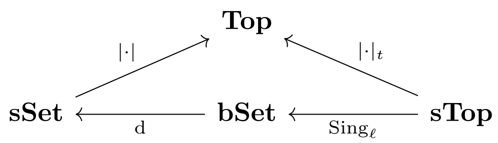
**Proposition 3.2.12.** [GJ09, Chapter IV, Section 1]. *For every $S \in$ **bSet** there is a natural isomorphism of simplicial sets $\mathrm{d}(S) \cong |S|_s$.*

Finally, we are ready to prove the desired weak homotopical factorization for good simplicial spaces:

**Theorem 3.2.13.** *If $X$ is a good simplicial space, then*

$$|\,\mathrm{d}(\mathrm{Sing}_\ell(X))| \simeq |X|_t,$$

*which is equivalent to the commutativity up to weak equivalence of the following diagram:*




*Proof.* By Proposition 3.2.12, it is known that $\mathrm{d}(S) \cong |S|_s$. Then, taking the geometric realization of this homeomorphism:

$$\begin{aligned}
|\,\mathrm{d}(S)| &\cong |(|S|_s)| \\
&= \left| \int^{n\in\Delta} S_{n,\bullet} \times \Delta[n] \right| \\
&\cong \int^{n\in\Delta} |S_{n,\bullet} \times \Delta[n]| && \text{(Using that } |\cdot| \text{ is a left adjoint functor)} \\
&\cong \int^{n\in\Delta} |S_{n,\bullet}| \times |\Delta[n]| && \text{(Using that } |\cdot| \text{ commutes with finite limits)} \\
&= \int^{n\in\Delta} (|S|_\ell)_n \times \Delta^n \\
&= |(|S|_\ell)|_t.
\end{aligned}$$

Then, there is a homeomorphism $\alpha : |\,\mathrm{d}(\mathrm{Sing}_\ell(X))| \cong |(|\,\mathrm{Sing}_\ell(X)|_\ell)|_t$.

On the other hand, using Lemma 3.2.9, we know that there is a natural levelwise weak homotopy equivalence $|\,\mathrm{Sing}_\ell(X)|_\ell \to X$. Also, by Lemma 3.2.8, $|\,\mathrm{Sing}_\ell(X)|_\ell$ is good. Finally, using Proposition 3.2.5, the previous levelwise weak homotopy equivalence induces $|(|\,\mathrm{Sing}_\ell(X)|_\ell)|_t \simeq |X|_t$, and composing with the homeomorphism $\alpha$ we obtain the desired weak equivalence. ☐

### 3.2.2 Diagonal simplicial nerve

As mentioned before, the diagonal simplicial nerve can be defined using a cosimplicial object in **sSet-Cat**, following the same pattern used to define the homotopy coherent nerve:

**Definition 3.2.14.** Let $[n] \in \Delta$. Define the *diagonal simplicial category* $\Delta^d[n] \in$ **sSet-Cat** as the simplicial category with:

- $\mathrm{Obj}(\Delta^d[n]) = [n] = \{0, \dots, n\}$.
- Morphisms and composition of $\Delta^d[n]$ are freely generated by the $n$-simplices $a_i \in \mathrm{Hom}(i-1, i)$ for all $i = 1, \dots, n$.

**Proposition 3.2.15.** *The map $[n] \mapsto \Delta^d[n]$ defines a covariant functor, i.e., $\Delta^d$ is a cosimplicial object in* **sSet-Cat***.*

*Proof.* Given a non-decreasing morphism $f : [m] \to [n]$, we only need to define the induced morphism $f^* : \Delta^d[m] \to \Delta^d[n]$. At the level of the objects $f^*$ is simply the application of $f$. Denote $a_i$ the generators of the morphisms of $\Delta^d[m]$ and $b_i$ the ones from $\Delta^d[n]$. Then we only need to define the image of $a_i \in \mathrm{Hom}(i-1, i)$ for $i = 1, \dots, m$. This can be defined as $f^*(a_i) = a_{f(i)} \in \mathrm{Hom}(f(i-1), f(i))$, which is an $m$-simplex contained in the $n$-simplex $b_{f(i)}$. Therefore, the map $[n] \mapsto \Delta^d[n]$ defines a covariant functor. □

Then, we can define a nerve and realization from this cosimplicial object. In this case, it turns out that we are only interested in the nerve, which follows Definition 1.4.4:

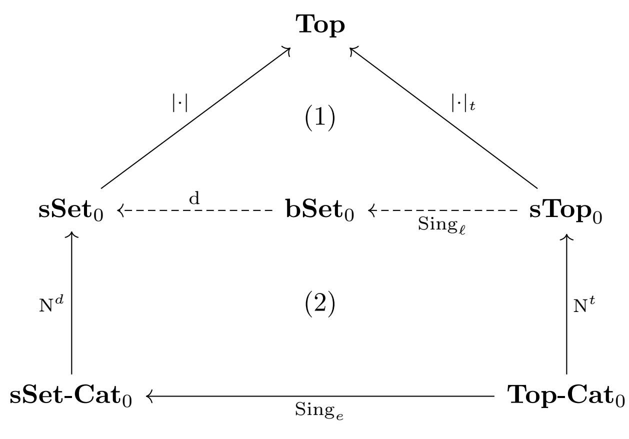
**Definition 3.2.16.** Define the *diagonal simplicial nerve* $\mathrm{N}^d$ : **sSet-Cat** $\to$ **sSet** as the following functor for any $\mathcal{C} \in$ **sSet-Cat**:

$$\mathrm{N}^d_n(\mathcal{C}) := \mathrm{N}^{\Delta^d}_n(\mathcal{C}) = \mathbf{sSet\text{-}Cat}(\Delta^d[n], \mathcal{C}).$$

Now that we have defined the diagonal simplicial nerve, we need a way to prove that it induces a functorial classifying space. Recall that we want to define the diagonal nerve classifying space as the functor

$$M \mapsto |\,\mathrm{N}^d(\mathrm{Sing}_e(\mathbb{D}(M)))|.$$

To prove that this functor realizes a classifying space, we will show that the following diagram commutes up to weak equivalence:




Observe that the triangle from (1) is equivalent to Theorem 3.2.13 proven in the last subsection. Thus, it is enough to prove that the rectangle (2) commutes up to isomorphism.

In order to prove the commutativity of (2), we need to study some more properties of the diagonal simplicial nerve. First, consider the characterization introduced above of the diagonal simplicial nerve as composition of functors, which appeared in the work of Dwyer and Kan [DK80]. To prove it, we need the functors presented before. The simplicial inclusion

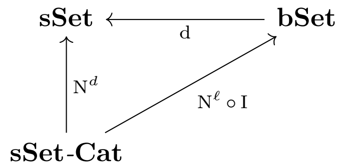
I has been defined in Remark 2.2.2, and the *levelwise nerve* $\mathrm{N}^\ell : \mathbf{sCat} \to \mathbf{bSet}$ can be defined as the functor which sends any $\mathcal{C}_\bullet \in \mathbf{sCat}$ to

$$(\mathrm{N}^\ell(\mathcal{C}_\bullet))_{n,m} = N_m(\mathcal{C}_n).$$

**Lemma 3.2.17.** *Let $\mathcal{C}$ be a simplicial category. Then*

$$\mathrm{N}^d(\mathcal{C}) = \mathrm{d}(\mathrm{N}^\ell(\mathrm{I}(\mathcal{C}))),$$

*which is equivalent to the commutativity of the following diagram:*




*Proof.* By definition, $\mathrm{N}^d_n(\mathcal{C}) = \mathbf{sSet\text{-}Cat}(\Delta^d[n], \mathcal{C})$. Because $\Delta^d[n]$ is determined by the $n$ generating $n$-simplices $a_i$, all the functors of $\mathrm{N}^d_n(\mathcal{C})$ are determined by the image of these simplices. Then, the images of the generating $n$-simplices $a_i$ are $n$ composable $n$-simplices of homsets of $\mathcal{C}$.

Now consider the right term. By the same argument of the previous proof, we know that $\mathrm{N}^\ell(\mathrm{I}(\mathcal{C}))$ is defined as a bisimplicial set which for every $n, m \in \mathbb{N}$ has a set $\mathrm{N}^\ell_n(\mathrm{I}_m(M))$ which contains all tuples of $n$ composable morphisms that are $m$-simplices of $\mathcal{C}$. Thus, taking the diagonal, we obtain a simplicial set with all $n$ composable $n$-simplices of homsets of $\mathcal{C}$ at level $n$, exactly the same as before. ☐

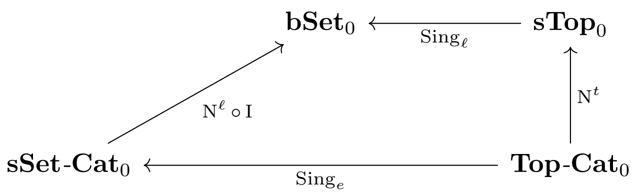
**Corollary 3.2.18.** *The diagonal simplicial nerve preserves weak equivalences.*

*Proof.* Let $\mathcal{C}$ and $\mathcal{D}$ be two simplicial categories with a weak equivalence between them $F : \mathcal{C} \xrightarrow{\sim} \mathcal{D}$. Then, we have weak equivalences of simplicial sets $\mathcal{C}(X, Y) \xrightarrow{\sim} \mathcal{D}(FX, FY)$ for each $X, Y \in \mathcal{C}$. Consider the bisimplicial sets $\mathrm{N}^\ell(\mathrm{I}(\mathcal{C}))$ and $\mathrm{N}^\ell(\mathrm{I}(\mathcal{D}))$. If we fix $n \in \mathbb{N}$ we obtain the levelwise simplicial sets $\mathrm{N}^\ell_n(\mathrm{I}(\mathcal{C}))$ and $\mathrm{N}^\ell_m(\mathrm{I}(\mathcal{C}))$. Then, $\mathrm{N}^\ell_n(\mathrm{I}(\mathcal{C}))$ is a simplicial set with all $n$ composable $n$-simplices of homsets of $\mathcal{C}$ at level $n$. Thus, taking compositions of weak equivalences between homsets of $\mathcal{C}$ and $\mathcal{D}$, it follows that there is a levelwise weak equivalence $\mathrm{N}^\ell_n(\mathrm{I}(\mathcal{C})) \xrightarrow{\sim} \mathrm{N}^\ell_m(\mathrm{I}(\mathcal{C}))$. Finally, by Proposition 3.2.11, this induces a weak equivalence between $\mathrm{N}^d(\mathcal{C}) = \mathrm{d}(\mathrm{N}^\ell(\mathrm{I}(\mathcal{C})))$ and $\mathrm{N}^d(\mathcal{D}) = \mathrm{d}(\mathrm{N}^\ell(\mathrm{I}(\mathcal{D})))$. ☐

**Lemma 3.2.19.** *Let $\mathcal{X}$ be a topological category with only one object. Then*

$$\mathrm{Sing}_\ell(\mathrm{N}^t(\mathcal{X})) \cong \mathrm{N}^\ell(\mathrm{I}(\mathrm{Sing}_e(\mathcal{X}))),$$

*which is equivalent to the commutativity up to isomorphism of the following diagram:*



*Proof.* Assume that $* \in \mathcal{X}$ is the only object of $\mathcal{X}$, and $M := \mathrm{Hom}(*, *)$. First we will compute the left side. $\mathrm{N}^t(\mathcal{X})$ is a simplicial space with $\mathrm{N}^t(\mathcal{X})_n = M^n$ and $\mathrm{N}^t(\mathcal{X})_0 = *$. Then, $\mathrm{Sing}_\ell(\mathrm{N}^t(\mathcal{X}))$ is a bisimplicial set, which for every $n, m \in \mathbb{N}$ has a set $\mathrm{Sing}_m(\mathrm{N}^t_n(\mathcal{X})) = \mathrm{Sing}_m(M^n)$ and $\mathrm{Sing}_m(\mathrm{N}^t_0(\mathcal{X})) = \mathrm{Sing}_m(*) = *$.

Now consider the right side. $\mathrm{Sing}_e(\mathcal{X})$ is a simplicial category with one object $*$ and $\mathrm{Hom}(*, *) = \mathrm{Sing}(M)$. Then, $\mathrm{I}(\mathrm{Sing}_e(\mathcal{X}))$ is a simplicial object in **Cat** such that for each $m \in \mathbb{N}$, $\mathrm{I}_m(\mathrm{Sing}_e(\mathcal{X}))$ is a category with only one object $*$ and morphisms $\mathrm{Hom}(*, *) = \mathrm{Sing}_m(M)$. Finally, $\mathrm{N}^\ell(\mathrm{I}(\mathrm{Sing}_e(\mathcal{X})))$ is defined as a bisimplicial set which for every $n, m \in \mathbb{N}$ has a set $\mathrm{N}^\ell_n(\mathrm{I}_m(\mathrm{Sing}_e(\mathcal{X}))) = \mathrm{N}^\ell_n(\mathrm{Sing}_m(M)) = (\mathrm{Sing}_m(M))^n$ and $\mathrm{N}^\ell_0(\mathrm{I}_m(\mathrm{Sing}_e(\mathcal{X}))) = \mathrm{N}^\ell_0(\mathrm{Sing}_m(M)) = *$.

Because Sing is a right adjoint functor, it preserves limits. In particular, we have an isomorphism

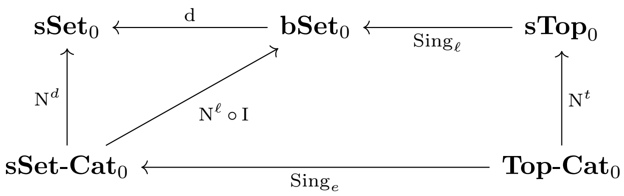
$$\mathrm{Sing}_m(\mathrm{N}^t_n(\mathcal{X})) = \mathrm{Sing}_m(M^n) \cong (\mathrm{Sing}_m(M))^n = \mathrm{N}^\ell_n(\mathrm{I}_m(\mathrm{Sing}_e(\mathcal{X}))). \qquad \square$$

Thus, we are finally ready to prove that the rectangle (2) commutes up to isomorphism, which is equivalent to the following theorem:

**Theorem 3.2.20.** *Let $\mathcal{X}$ be a topological category with only one object. Then*

$$\mathrm{d}(\mathrm{Sing}_\ell(\mathrm{N}^t(\mathcal{X}))) \cong \mathrm{N}^d(\mathrm{Sing}_e(\mathcal{X})).$$

*Proof.* Finally, we just need to join the two previous lemmas. Recall that the theorem is equivalent to proving a commuting rectangle, which we can divide in the two commutative diagrams from the lemmas:




Let $\mathcal{X}$ be a topological category with only one object. Then, by Lemma 3.2.19,

$$\mathrm{Sing}_\ell(\mathrm{N}^t(\mathcal{X})) \cong \mathrm{N}^\ell(\mathrm{I}(\mathrm{Sing}_e(\mathcal{X}))).$$

Now, we can apply the d functor to obtain an isomorphism of simplicial sets

$$\mathrm{d}(\mathrm{Sing}_\ell(\mathrm{N}^t(\mathcal{X}))) \cong \mathrm{d}(\mathrm{N}^\ell(\mathrm{I}(\mathrm{Sing}_e(\mathcal{X})))).$$

Finally, by Lemma 3.2.17 we know that

$$\mathrm{N}^d(\mathrm{Sing}_e(\mathcal{X})) = \mathrm{d}(\mathrm{N}^\ell(\mathrm{I}(\mathrm{Sing}_e(\mathcal{X})))),$$

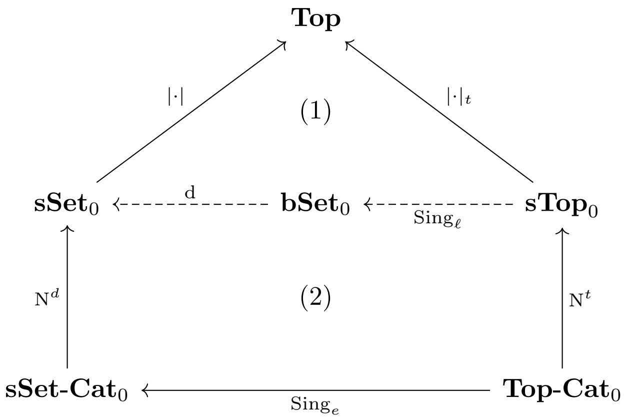
which directly implies the desired isomorphism. $\square$

**Corollary 3.2.21.** *Let $M$ be a well-pointed group-like topological monoid. Then*

$$|\,\mathrm{N}^d(\mathrm{Sing}_e(\mathbb{D}\,M))| \simeq |\,\mathrm{N}^t(\mathbb{D}\,M)|_t = \mathrm{B}(M).$$

*i.e., the functor $M \mapsto |\,\mathrm{N}^d(\mathrm{Sing}_e(\mathbb{D}\,M))|$ is a classifying space of $M$.*

*Proof.* Proving this fact is equivalent to showing that the exterior arrows of the following diagram commute up to weak equivalence:




Observe that, by Theorem 3.2.13, we know that the upper triangle (1) commutes up to weak equivalence. On the other hand, proving the commutativity up to isomorphism of the bottom rectangle (2) is equivalent to Theorem 3.2.20. Then, the desired result follows directly thanks to $|\cdot|$ preserving weak equivalences. ☐

## 3.3 Homotopy coherent nerve model

In this section, we want to finally prove that the functor $M \mapsto |\,\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathbb{D}(M)))|$ is a classifying space of $M$. Observe that the classifying space presented in the previous section has the same construction, only replacing the homotopy coherent nerve with the diagonal simplicial nerve. Then, it will be enough to prove that there is a weak equivalence between the diagonal simplicial nerve and the homotopy coherent nerve $\mathrm{N}^{\mathfrak{R}}(\mathcal{C}) \simeq \mathrm{N}^d(\mathcal{C})$ for every fibrant simplicial category $\mathcal{C}$ with $\mathrm{h}\mathcal{C}$ being a groupoid.

The existence of the weak equivalence between the diagonal simplicial nerve and the homotopy coherent nerve is inspired by the work of Hinich [Hin07]. In the following subsection, another simplicial nerve will be introduced: the *total simplicial nerve*. A simplicial groupoid is a simplicial category which, if considered as a simplicial object in **Cat**, is a groupoid at each one of the levels. Considering only simplicial groupoids, we prove the existence of a weak equivalence from the diagonal simplicial nerve to the total simplicial nerve, and another one from the total simplicial nerve to the homotopy coherent nerve. In the second subsection, we generalize this result from simplicial groupoids to fibrant simplicial category with the homotopy category being a groupoid. This generalization is achieved by using simplicial localization as defined by Dwyer and Kan [DK80].

### 3.3.1 Total simplicial nerve

We begin with the definition of the total simplicial nerve using a different cosimplicial object, as we have done with the two previous nerve functors:

**Definition 3.3.1.** The *total cosimplicial object* $\Delta^T[n] \in$ **sSet-Cat** is defined as follows:

- $\mathrm{Obj}(\Delta^T[n]) = [n] = \{0, \ldots, n\}$.

- Morphisms and composition of $\Delta^T[n]$ are freely generated by $(n-i)$-simplices $g_i \in \mathrm{Hom}(i-1,i)$ for $i = 1, \ldots, n$.

**Proposition 3.3.2.** *The map $[n] \mapsto \Delta^T[n]$ defines a covariant functor, i.e., $\Delta^T$ is a cosimplicial object in* **sSet-Cat**.

*Proof.* We will prove the cosimplicial structure defining the image of the generating morphisms of the $\Delta$ category. First, the map $\delta_i^n : [n-1] \to [n]$ has as image $\partial_i^n : \Delta^T[n-1] \to \Delta^T[n]$ which for any generating morphism $g_j \in \mathrm{Hom}(j-1,j)$ is defined by

$$\partial_i^n(g_j) = \begin{cases} d_{n-1-j}^{n-j}(g_j) & \text{if } i = n-1 \\ d_{i-j}^{n-j}(g_j) & \text{if } i \neq n-1 \text{ and } j < i \\ g_{i+1} \circ d_0^{n-i}(g_i) & \text{if } i \neq n-1 \text{ and } j = i \\ g_{j+1} & \text{if } i \neq n-1 \text{ and } j > i. \end{cases}$$

On the other hand, the map $\sigma_i^n : [n] \to [n-1]$ has as image $\omega_i^n : \Delta^T[n] \to \Delta^T[n-1]$, which for any generating morphism $g_j \in \mathrm{Hom}(j-1,j)$ is defined by

$$\omega_i^n(g_j) = \begin{cases} s_{i-j}^{n-j}(g_j) & \text{if } j < i+1 \\ \mathrm{Id}_j & \text{if } j = i+1 \\ g_{j-1} & \text{if } j > i+1. \end{cases}$$

□

Thus, we can define a nerve and realization from the cosimplicial objects $\Delta^T[n]$ following the construction in Definition 1.4.4. We will only use the nerve functor:

**Definition 3.3.3.** The *total simplicial nerve* $\mathrm{N}^T$ : **sSet-Cat** $\to$ **sSet** is defined for every $\mathcal{C} \in$ **sSet-Cat** as

$$\mathrm{N}_n^T(\mathcal{C}) := \mathrm{N}_n^{\Delta^T}(\mathcal{C}) = \mathbf{sSet\text{-}Cat}(\Delta^T[n], \mathcal{C}).$$

This simplicial nerve, similarly to the diagonal simplicial nerve, has a characterization using well-known functors. The functor used here is the *classifying complex* $\overline{W}$ : **sGrp** $\to$ **sSet** defined on every simplicial group $\mathcal{G}$ as

$$\overline{W}_n(\mathcal{G}) = \mathcal{G}_{n-1} \times \mathcal{G}_{n-2} \times \cdots \times \mathcal{G}_0.$$

Then, we can prove the following characterization for simplicial groups:

**Proposition 3.3.4.** *If $\mathcal{G}$ is a simplicial group,* $\mathrm{N}_n^T(\mathbb{D}(\mathcal{G})) = \overline{W}(\mathcal{G})$.

*Proof.* By definition, $\overline{W}_n(\mathcal{G}) = \mathcal{G}_{n-1} \times \mathcal{G}_{n-2} \times \cdots \times \mathcal{G}_0$. On the other hand, consider the simplicial category $\mathbb{D}(\mathcal{G})$, which has one object $*$ and morphisms $\mathrm{Hom}(*,*)_n = \mathcal{G}_n$. Then, for each $n \in \mathbb{N}$, the total simplicial nerve is defined by all the functors from $\Delta^T[n]$ to $\mathbb{D}(\mathcal{G})$. These functors are determined by the image of the generating morphisms of $\Delta^T[n]$. Thus, each functor is determined by choosing one element from each set $\mathcal{G}_{n-1}, \mathcal{G}_{n-2}, \ldots$, and $\mathcal{G}_0$, i.e.,

$$\mathrm{N}_n^T(\mathbb{D}(\mathcal{G})) = \mathcal{G}_{n-1} \times \mathcal{G}_{n-2} \times \cdots \times \mathcal{G}_0 = \overline{W}_n(\mathcal{G}).$$

□

Now, we want to prove the weak equivalences between the three previously defined simplicial nerves restricted to simplicial groupoids. First, define the following map of cosimplicial objects:

$$\alpha : \Delta^T \to \Delta^d$$

such that for each $n \in \mathbb{N}$ it defines the identity map on objects, and it is defined on the generating morphisms as

$$\alpha_n(g_i) = (d_0^n)^i f_i.$$

**Proposition 3.3.5.** *Let $\mathcal{C} \in$ **sSet-Cat**. If $\mathcal{C}$ is simplicial groupoid, then there is weak equivalence* $\mathrm{N}^d(\mathcal{C}) \simeq \mathrm{N}^T(\mathcal{C})$.

*Proof.* First, observe that the induced map is:

$$\begin{array}{cccc} \alpha^* = \mathrm{Hom}(\alpha, \mathcal{C}) : & \mathrm{N}^d(\mathcal{C}) & \longrightarrow & \mathrm{N}^T(\mathcal{C}) \\ & (f_1, f_2, \dots, f_n) & \longmapsto & (d_0^n f_1, (d_0^n)^2 f_2, \dots, (d_0^n)^n f_n) \end{array}$$

Let us assume that $\mathcal{C}$ is connected, otherwise this argument will apply component-wise. Because $\mathcal{C}$ is a connected simplicial groupoid, it is equivalent to a simplicial group $\mathcal{G}$ generated by one of the objects $*$ and the homset $\mathrm{Hom}(*, *)$. Because $\mathcal{G}$ is a simplicial group, to prove that $\alpha^*$ is a weak equivalence we only need to show that it has the RLP with respect to the maps $\partial\Delta[n] \to \Delta[n]$.

First, we need to choose suitable maps. Pick the map $\Delta[n] \to \mathrm{N}_n^T(\mathcal{G})$ with image $g = (g_1, \dots, g_n)$ and $g_i \in \mathrm{Hom}_{n-i}(i-1, i)$. Also, the map $\partial\Delta[n] \to \mathrm{N}_{n-1}^d(\mathcal{G})$, which for each $0 \leq i \leq n-1$ has as image of the $i$-th face of $\Delta[n]$ the tuple of $(n-1)$-simplices $x^i = (x_1^i, \dots, x_{n-1}^i)$. Then, we want to find a map $\Delta[n] \to \mathrm{N}_n^d(\mathcal{G})$, which is equivalent to choosing a tuple $f = (f_1, \dots, f_n)$ which commutes with the other maps.

The two maps defined satisfy the relations $d_i^{n-1}(x^k) = d_{k-1}^{n-1}(x^i)$ for all $i < k$ and $\alpha^*(x^i) = d_i^n(g)$. The desired map $f$ has to satisfy $d_i^n f = x^i$ and $\alpha^*(f) = g$, which is equivalent to the three following conditions:

1. $(d_0^n)^j(f_j) = g_j$,
2. $d_i^n(f_j) = \begin{cases} x_j^i & \text{if } i \geq j+1 \\ x_{j-1}^i & \text{if } i \leq j-2, \end{cases}$
3. $d_{j-1}^n(f_j) \circ d_{j-1}^n(f_{j-1}) = x_{j-1}^{j-1}$.

Finally, we can deduce the values of $f_j$ by induction. Imposing the desired conditions and using that $\mathcal{G}$ is fibrant, we can easily obtain $f_1$. For the rest, we want to construct $f_j$ assuming $f_i$ for all $i < j$. The technical details can be found in [Hin01, Section A.5.1]. Therefore, we have proven that $\alpha^*$ is a weak equivalence. ☐

The previous weak equivalence allows us to write an unusual proof of the geometric realization of the classifying complex $\overline{W}$ having the homotopy type of the Milgram classifying space for every simplicial group. The classifying complex realizing the classifying space is a well-known fact in the literature, and similar factorizations can be found in [CR05] and [Ste11].

**Corollary 3.3.6.** *Let $G$ be a well-pointed topological group. Then*

$$\mathrm{B}(M) = |\,\widetilde{\mathrm{B}}(G)|_t \simeq |\overline{W}(\mathrm{Sing}(G))|,$$

*i.e., the functor $G \mapsto |\overline{W}(\mathrm{Sing}(G))|$ is a classifying space of $G$.*

*Proof.* First observe that any well-pointed topological group is a well-pointed group-like monoid, which implies that $\mathrm{B}(M)$ is well-defined.

On the other hand, observe that $\mathrm{Sing}(G)$ is a simplicial group. Hence, $\mathrm{Sing}(G)$ can be considered as a simplicial category with one object $\mathbb{D}(\mathrm{Sing}(G))$. By Proposition 3.3.4, we know that

$$\overline{W}(\mathrm{Sing}(G)) = \mathrm{N}^T(\mathbb{D}(\mathrm{Sing}(G))) = \mathrm{N}^T(\mathrm{Sing}_e(\mathbb{D}(G))).$$

Then, we can use Proposition 3.3.5, obtaining

$$\mathrm{N}^d(\mathrm{Sing}_e(\mathbb{D}(G))) \simeq \mathrm{N}^T(\mathrm{Sing}_e(\mathbb{D}(G))) = \overline{W}(\mathrm{Sing}(G)).$$

Finally, by Corollary 3.2.21, because the geometric realization of the diagonal nerve is a model of the classifying space, we obtain the desired result:

$$\mathrm{B}(M) \simeq |\,\mathrm{N}^d(\mathrm{Sing}_e(\mathbb{D}(G)))| \simeq |\,\mathrm{N}^T(\mathrm{Sing}_e(\mathbb{D}(G)))| = |\overline{W}(\mathrm{Sing}(G))|. \qquad \square$$

Now we will construct another map between simplicial nerves. This time we want to prove that there exists a unique map between the simplicial objects $\Delta^{\Re}$ and $\Delta^T$.

**Lemma 3.3.7.** *There exists a unique map between the simplicial objects*

$$\beta : \Delta^{\Re} \to \Delta^T$$

*which is bijective on objects.*

*Proof.* Define $\beta_n : \Delta^{\Re}[n] \to \Delta^T[n]$, the component maps of $\beta$. Because the simplicial sets of morphisms of each $\Delta^{\Re}[n]$ and $\Delta^T[n]$ are nerves of posets, the map $\beta_n$ is uniquely defined by the restriction on 0-simplices, which must be monotone. We already know that the only indecomposable 0-simplices of $\Delta^{\Re}[n]$ are the sets $\{a, b\} \in \mathrm{Hom}(a, b)$ for every $0 \le a < b \le n$. Then, to define $\beta_n$, we only need to give the image of those 0-simplices in a way that $\beta_n$ is monotone.

First consider the case $n = 1$. The simplicial category $\Delta^{\Re}[1]$ has two objects and one non-trivial homset, $\mathrm{Hom}(0, 1)$, which has one non-degenerate indecomposable 0-simplex $\{0, 1\}$. On the other hand, $\Delta^T[1]$ has two objects and also one non-trivial homset, $\mathrm{Hom}(0, 1)$, which also contains a generating 0-simplex $g_0$. Then, we must define $\beta_1(0) = 0$, $\beta_1(1) = 1$ and $\beta_1(\{0, 1\}) = g_1$.

Now consider the map $\varphi : [1] \to [n]$ defined by $\varphi(0) = a$ and $\varphi(1) = b$. We can reduce this monotone map to a composition of faces:

$$\varphi = \partial_n^{b+1} \circ \partial_{n-1}^{b+1} \circ \cdots \circ \partial_{b+1}^{b+1} \circ \partial_b^{a+1} \circ \cdots \circ \partial_{a+2}^{a+1} \circ \partial_{a+1}^0 \circ \cdots \circ \partial_2^0. \tag{3.3}$$

Because $\beta$ is a map between cosimplicial objects, the induced maps from $\varphi$ to each cosimplicial object, denoted $\varphi^*_{\Re} : \Delta^{\Re}[1] \to \Delta^{\Re}[n]$ and $\varphi^*_T : \Delta^T[1] \to \Delta^T[n]$, must make the following diagram commutative:

$$\begin{array}{ccc} \Delta^{\Re}[1] & \xrightarrow{\beta_n} & \Delta^T[1] \\ \downarrow{\scriptstyle \varphi^*_{\Re}} & & \downarrow{\scriptstyle \varphi^*_T} \\ \Delta^{\Re}[n] & \xrightarrow{\beta_n} & \Delta^T[n] \end{array}$$

Then, we know that $\varphi^*_{\Re}(\{0, 1\}) = \{\varphi(0), \varphi(1)\} = \{a, b\}$, and using the commutativity, we obtain

$$\beta_n(\{a, b\}) = \beta_n(\varphi^*_{\Re}(\{0, 1\})) = \varphi^*_T(\beta_1(\{0, 1\})) = \varphi^*_T(g_1).$$

Using the formulas of Proposition 3.3.2 and Equation 3.3, we can compute this last term as

$$\beta_n(\{a, b\}) = \varphi^*_T(g_1) = d_1^{n-b}(g_b) \circ d_1^{n-b}(d_0(g_{b-1})) \circ \cdots \circ d_1^{n-b}(d_0^{b-a-1}(g_{a+1})).$$

Finally, it is clear that this definition is monotone, because using $d_1 x < d_0 x$ and the previous equation, we have $\beta_n(\{b, c\})\beta_n(\{a, b\}) < \beta_n(\{a, c\})$. $\square$

Finally, we want to see that the induced map between the simplicial nerves is also a weak equivalence for simplicial groupoids:

**Proposition 3.3.8.** *Let* $\mathcal{C} \in$ **sSet-Cat**. *If* $\mathcal{C}$ *is simplicial groupoid, then there is weak equivalence* $\mathrm{N}^T(\mathcal{C}) \simeq \mathrm{N}^{\Re}(\mathcal{C})$.

*Proof.* First assume that $\mathcal{C}$ is connected, otherwise this argument will apply component-wise. Because $\mathcal{C}$ is a connected simplicial groupoid, it is equivalent to a simplicial group generated by one of the objects $*$ and the homset $\mathrm{Hom}(*, *)$. By Proposition 3.3.4 and the fact that $\overline{W}$ models the classifying space of a simplicial group, we know that

$$\pi_n(\mathrm{N}^T(\mathcal{C})) = \pi_n(\overline{W}(\mathrm{Hom}(*, *))) = \pi_{n-1}(\mathrm{Hom}(*, *)).$$

Then, proving that the desired map is a weak equivalence corresponds to proving that $\pi_n(\mathrm{N}^{\Re}(\mathcal{C})) = \pi_{n-1}(\mathrm{Hom}(*, *))$.

Thanks to Proposition 2.2.20, we know that $\mathrm{N}^{\Re}(\mathcal{C})$ is a Kan complex. Let $\mathcal{S}^n = \Delta[n]/\partial\Delta[n]$ and $\mathcal{D}^{n+1} = \Delta[n+1]/\bigcup_{i>0} \partial^i(\Delta[n])$. The face map induces an embedding $\mathcal{S}^n \to \mathcal{D}^{n+1}$, which, using that $\mathrm{N}^{\Re}(\mathcal{C})$ is a Kan complex, implies that

$$\begin{aligned}\pi_n(\mathrm{N}^{\Re}(\mathcal{C})) &= \mathrm{Hom}(\mathcal{S}^n, \mathrm{N}^{\Re}(\mathcal{C}))/\mathrm{Hom}(\mathcal{D}^{n+1}, \mathrm{N}^{\Re}(\mathcal{C})) \\ &= \mathrm{Hom}(\Delta^{\Re}(\mathcal{S}^n), \mathcal{C})/\mathrm{Hom}(\Delta^{\Re}(\mathcal{D}^{n+1}), \mathcal{C}).\end{aligned}$$

By Example 2.2.14, we know that $\Delta^{\Re}(\mathcal{S}^n)$ is a simplicial category with one object and automorphism set equal to $\mathcal{S}^1 \wedge \cdots \wedge \mathcal{S}^1$ ($n-1$ times). Furthermore, by Example 2.2.15, we know that $\Delta^{\Re}(\mathcal{D}^{n+1})$ is a simplicial category with one object and automorphism set equal to $I \wedge \mathcal{S}^1 \wedge \cdots \wedge \mathcal{S}^1$ ($n-1$ times). Observe that $\mathcal{S}^1 \wedge \cdots \wedge \mathcal{S}^1$ ($n-1$ times) is homotopy equivalent to $\mathcal{S}^{n-1}$ and $I \wedge \mathcal{S}^1 \wedge \cdots \wedge \mathcal{S}^1$ ($n-1$ times) is homotopy equivalent to $\mathcal{D}^n$. Also, remember that each simplicial groupoid is in particular a fibrant simplicial category. Thus, using that $\mathrm{Hom}(*, *)$ is a Kan complex because $\mathcal{C}$ is a fibrant simplicial category, we have $\pi_n(\mathrm{N}^{\Re}(\mathcal{C})) = \pi_{n-1}(\mathrm{Hom}(*, *))$, as we wanted to prove. □

### 3.3.2 Simplicial localization

In this subsection, we want to extend the weak equivalence between nerves of simplicial groupoids presented previously. We will be able to extend it to fibrant simplicial categories whose homotopy category is a groupoid. This will be done using the simplicial localization as defined by Dwyer and Kan [DK80].

Let **bSet-Cat** be the category of categories enriched in bisimplicial sets. First, we need to define simplicial localization, and gather the necessary properties of this construction. To introduce the simplicial localization, we need to define the following:

**Definition 3.3.9.** (i) Given a category $\mathcal{C}$, the *free category* on $\mathcal{C}$ is a category $\mathrm{F}_1\,\mathcal{C}$ which has the same objects as $\mathcal{C}$ and has one generator for each non-identity map. Then, define the *free resolution* on $\mathcal{C}$ as the simplicial category $\mathrm{F}_*\,\mathcal{C}$ which has the same objects as $\mathcal{C}$ and as $n$-simplices it has the morphisms of $\mathrm{F}_n\,\mathcal{C} := \mathrm{F}_1^n\,\mathcal{C}$. In the case of simplicial categories, the free resolution is in fact a bisimplicially enriched category, taking the free resolution at each level.

(ii) Define the *simplicial diagonal* $\mathrm{d}_e$ : **bSet-Cat** $\to$ **sSet-Cat** as taking the diagonal of each homset.

**Definition 3.3.10.** Let $\mathcal{C}$ be a simplicial category and $\mathcal{W}$ a subcategory. The *simplicial localization* of $\mathcal{C}$ with respect to $\mathcal{W}$ is a simplicial category defined by

$$\mathcal{L}(\mathcal{C},\mathcal{W}) := \mathrm{d}_e(\mathrm{F}_*\,\mathcal{C}[(\mathrm{F}_*\,\mathcal{W})^{-1}]).$$

**Lemma 3.3.11.** *Let $\mathcal{C}$ be a simplicial category. Then $\mathcal{L}(\mathcal{C},\mathcal{C})$ is a simplicial groupoid.*

*Proof.* Recall that $\mathcal{L}(\mathcal{C},\mathcal{C}) = \mathrm{d}_e(\mathrm{F}_*\,\mathcal{C}[(\mathrm{F}_*\,\mathcal{C})^{-1}])$. Then, for each $n \in \mathbb{N}$, the induced category of the objects of $\mathcal{L}(\mathcal{C},\mathcal{C})$ and the $n$-simplices of $\mathcal{L}(\mathcal{C},\mathcal{C})$ is the full localization of the $n$-th free category generated by the $n$-simplex of $\mathcal{C}$. Therefore, all $n$-morphisms are invertible. □

**Lemma 3.3.12.** *If $\mathcal{C}$ is a cofibrant simplicial category with $\mathrm{h}\mathcal{C}$ a groupoid, then the localization map $\mathcal{C} \to \mathcal{L}(\mathcal{C},\mathcal{C})$ is a weak equivalence.*

*Proof.* By Theorem 2.2.8 and using that $\mathcal{C}$ is cofibrant simplicial groupoid, we know that there exists a weak equivalence $\mathcal{C} \to \mathrm{F}_n\,\mathcal{C}$, which arises from the fact that cofibrant objects are retracts from free objects. Observe that the formal inversion morphism $\mathrm{F}_n\,\mathcal{C} \to \mathrm{F}_n\,\mathcal{C}[(\mathrm{F}_n\,\mathcal{C})^{-1}]$ is also a weak equivalence thanks to $\mathcal{C}$ being a simplicial category with $\mathrm{h}\mathcal{C}$ a groupoid. Then, composing the two maps and taking the diagonal, we obtain the desired weak equivalence $\mathcal{C} \to \mathcal{L}(\mathcal{C},\mathcal{C})$. □

Consider the composition of the two maps between cosimplicial objects defined in the previous subsection, and denoted by

$$\tau : \Delta^{\Re} \to \Delta^{d}. \tag{3.4}$$

Using all the previous results, we are ready to prove the following theorem, relating the homotopy coherent nerve with the diagonal simplicial nerve. Using this theorem and Corollary 3.2.21, we can finally prove that the homotopy coherent nerve gives a functorial classifying space of a topological monoid:

**Theorem 3.3.13.** *For every fibrant simplicial category $\mathcal{C}$ with $\mathrm{h}\mathcal{C}$ a groupoid, the induced map $\mathrm{N}^d(\mathcal{C}) \to \mathrm{N}^{\Re}(\mathcal{C})$ is a homotopy equivalence of simplicial sets.*

*Proof.* First, take a cofibrant replacement of $\mathcal{C}$, $Q_{\mathcal{C}} : \widetilde{\mathcal{C}} \xrightarrow{\sim} \mathcal{C}$. Because $\mathcal{C}$ is fibrant, $\widetilde{\mathcal{C}}$ will be also fibrant. Then, there are associated maps

$$\mathrm{N}^d(Q_{\mathcal{C}}) = \mathrm{Hom}(\Delta^d, Q_{\mathcal{C}}) : \mathrm{N}^d(\widetilde{\mathcal{C}}) \to \mathrm{N}^d(\mathcal{C}), \text{ and}$$
$$\mathrm{N}^{\Re}(Q_{\mathcal{C}}) = \mathrm{Hom}(\Delta^{\Re}, Q_{\mathcal{C}}) : \mathrm{N}^{\Re}(\widetilde{\mathcal{C}}) \to \mathrm{N}^{\Re}(\mathcal{C}).$$

By Corollary 3.2.18 and $Q_{\mathcal{C}}$ being a weak equivalence, $\mathrm{N}^d(Q_{\mathcal{C}})$ is a weak equivalence. Then, thanks to Proposition 2.2.19, $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ being fibrant simplicial categories, and $Q_{\mathcal{C}}$ being a weak equivalence, we have that $\mathrm{N}^{\Re}(Q_{\mathcal{C}})$ is a weak equivalence too.

Because $\widetilde{\mathcal{C}}$ is a cofibrant simplicial category with $\mathrm{h}\widetilde{\mathcal{C}}$ a groupoid, Lemma 3.3.12 proves that there exists a weak equivalence $\ell_{\widetilde{\mathcal{C}}} : \widetilde{\mathcal{C}} \xrightarrow{\sim} \mathcal{L}(\widetilde{\mathcal{C}},\widetilde{\mathcal{C}})$. This map $\ell_{\widetilde{\mathcal{C}}}$ has the associated morphisms

$$\mathrm{N}^d(\ell_{\widetilde{\mathcal{C}}}) = \mathrm{Hom}(\Delta^d, \ell_{\widetilde{\mathcal{C}}}) : \mathrm{N}^d(\widetilde{\mathcal{C}}) \to \mathrm{N}^d(\mathcal{L}(\widetilde{\mathcal{C}},\widetilde{\mathcal{C}})), \text{ and}$$
$$\mathrm{N}^{\Re}(\ell_{\widetilde{\mathcal{C}}}) = \mathrm{Hom}(\Delta^{\Re}, \ell_{\widetilde{\mathcal{C}}}) : \mathrm{N}^{\Re}(\widetilde{\mathcal{C}}) \to \mathrm{N}^{\Re}(\mathcal{L}(\widetilde{\mathcal{C}},\widetilde{\mathcal{C}})).$$

Observe that because of Lemma 3.3.11, we know that $\mathcal{L}(\widetilde{\mathcal{C}},\widetilde{\mathcal{C}})$ is a simplicial groupoid, which in particular is also a fibrant simplicial category. Then, by the same arguments as before, $\mathrm{N}^d(\ell_{\widetilde{\mathcal{C}}})$ and $\mathrm{N}^{\Re}(\ell_{\widetilde{\mathcal{C}}})$ are weak equivalences.

On the other hand, consider the map of Equation 3.4, which for every simplicial category $\mathcal{D}$ induces

$$\tau^*_{\mathcal{D}} = \mathrm{Hom}(\tau, \mathcal{D}) : \mathrm{N}^d(\mathcal{D}) \to \mathrm{N}^{\mathfrak{R}}(\mathcal{D}).$$

Thanks to Proposition 3.3.8 and Proposition 3.3.5, if $\mathcal{D}$ is a simplicial groupoid, $\tau^*_{\mathcal{D}}$ is a weak equivalence. Then, using Lemma 3.3.11, we know that $\tau^*_{\mathcal{L}(\widetilde{\mathcal{C}},\widetilde{\mathcal{C}})}$ is a weak equivalence.

Finally, by naturality of the Hom bifunctor, we know that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{N}^d(\mathcal{L}(\widetilde{\mathcal{C}},\widetilde{\mathcal{C}})) & \xrightarrow{\tau^*_{\mathcal{L}(\widetilde{\mathcal{C}},\widetilde{\mathcal{C}})}} & \mathrm{N}^{\mathfrak{R}}(\mathcal{L}(\widetilde{\mathcal{C}},\widetilde{\mathcal{C}})) \\
\uparrow {\scriptstyle \mathrm{N}^d(\ell_{\widetilde{\mathcal{C}}})} & & \uparrow {\scriptstyle \mathrm{N}^{\mathfrak{R}}(\ell_{\widetilde{\mathcal{C}}})} \\
\mathrm{N}^d(\widetilde{\mathcal{C}}) & \overset{\tau^*_{\widetilde{\mathcal{C}}}}{\dashrightarrow} & \mathrm{N}^{\mathfrak{R}}(\widetilde{\mathcal{C}}) \\
\downarrow {\scriptstyle \mathrm{N}^d(Q_{\mathcal{C}})} & & \downarrow {\scriptstyle \mathrm{N}^{\mathfrak{R}}(Q_{\mathcal{C}})} \\
\mathrm{N}^d(\mathcal{C}) & \overset{\tau^*_{\mathcal{C}}}{\dashrightarrow} & \mathrm{N}^{\mathfrak{R}}(\mathcal{C})
\end{array}$$

Observe that by the previous arguments all the solid arrows are weak equivalences. Thus, using the commutativity and the two-out-of-three property, it follows directly that the upper dotted arrow is a weak equivalence, and using this fact, we can prove by the same argument that the bottom dotted arrow is a weak equivalence. □

**Corollary 3.3.14.** *Let $M$ be a well-pointed group-like topological monoid. Then*

$$\mathrm{B}(M) = |\,\widetilde{\mathrm{B}}(M)|_t \simeq |\,\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathbb{D}\,M))|,$$

*i.e., the functor $M \mapsto |\,\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathbb{D}\,M))|$ is a classifying space of $M$.*

*Proof.* By Corollary 3.2.21, we know that

$$\mathrm{B}(M) = |\,\widetilde{\mathrm{B}}(M)|_t \simeq |\,\mathrm{N}^d(\mathrm{Sing}_e(\mathbb{D}\,M))|.$$

On the other hand, by Theorem 3.3.13, $\mathrm{N}^d(\mathcal{C}) \simeq \mathrm{N}^{\mathfrak{R}}(\mathcal{C})$ for every weak fibrant simplicial groupoid $\mathcal{C}$. We know that $\mathrm{Sing}_e(\mathbb{D}(M))$ is always a fibrant simplicial category. Also, because $M$ is a group-like topological monoid, $\mathrm{Sing}_e(\mathbb{D}(M))$ is a fibrant simplicial category with $\mathrm{h}(\mathrm{Sing}_e(\mathbb{D}(M)))$ a groupoid. Thus, we have

$$\mathrm{N}^d(\mathrm{Sing}_e(\mathbb{D}(M))) \simeq \mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathbb{D}(M))).$$

Finally, we know that $|\cdot|$ preserves weak equivalences, which implies the desired result. □

Thus, using previous results, we can deduce the proof of Main Theorem. This theorem is basically a direct consequence of the previous functorial classifying space, together with particular properties of the Moore path loop space:

**Main Theorem.** *Let $(X, x)$ be a path-connected well-pointed topological space. The topological space $|\,\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathbb{D}\,\Omega^M_x(X)))|$ is a classifying space for $\Omega^M_x(X)$, and as a consequence, there is a natural weak homotopy equivalence*

$$|\,\mathrm{N}^{\mathfrak{R}}(\mathrm{Sing}_e(\mathbb{D}\,\Omega^M_x X))| \simeq X.$$

*Proof.* Follows directly from the previous corollary applied to $\Omega^M_x X$ and Proposition 3.1.6. □

# Chapter 4

# Application to homotopy type theory

As mentioned in the Introduction, the objective of this last chapter is to assess whether the model of the fundamental $\infty$-groupoid as a Moore path category can help to the interpretation of results from homotopy type theory. The first section of this chapter introduces the basics of homotopy type theory. In the second one, we offer a review of the first homotopical models of homotopy type theory, and explain the relation between types and $\infty$-groupoids. In the last section, we study whether Moore path categories are a useful tool for proving that the type-theoretic definition of well-known topological spaces as higher inductive types corresponds to the $\infty$-groupoids of these spaces.

## 4.1 Homotopy type theory

A *Martin-Löf type theory* is a deductive system based on judgements and rules of inference. The judgements are the "propositions" of the deductive system, and the rules of inference are used to derive new judgements from previous ones. Consequently, we can construct a derivation tree "to prove" some judgement by indicating which rules of inference have been used at each step.

Any Martin-Löf type theory has four different kinds of judgements. The first one is the type declaration, denoted by $\vdash A$ type, which means that $A$ is a type. Whenever we know that $A$ is a type, we can introduce an element $a$ of type $A$ with the judgement of term declaration $\vdash a : A$. The other two are equalities: the equality between terms $\vdash a = b$, and the equality between types $\vdash A = B$.

Observe that judgements begin with the symbol $\vdash$, commonly called turnstile. The turnstile divides any judgement in two parts: the left side is the context, and the right side the outcome. The context is used to introduce type dependency, which is one of the main properties of Martin-Löf type theory, and is composed of a comma separated list of term declarations. Notice that the previous judgements have an empty context, but could be also found with any non-empty context. For instance, using type dependency, we can introduce a family $B$ indexed by $A$ as the judgement

$$x : A \vdash B(x) \text{ type},$$

which means that for each element $a : A$, $B(a)$ is a type.

On the other hand, we have the rules of inference. Any rule of inference has a finite set of judgements as assumptions, and a unique judgement as conclusion. For example, the rule with assumptions $\vdash a = b$ and $\vdash b = c$, and conclusion $\vdash a = c$, is denoted by

$$\frac{\vdash a = b \qquad \vdash b = c}{\vdash a = c}.$$

To define a new type, we need to specify a set of rules that determine the behavior of the type with respect to the rest of the type theory. This set of rules has a structure, where each rule must have a clear predefined goal. Here, the definition of the type of functions is used as an illustration. Let $A$ and $B$ be two types. Then, we define the type of functions $A \to B$ with the following rules:

- *Formation rule*: Give the conditions required to form a type.

$$\frac{\vdash A \text{ type} \qquad \vdash B \text{ type}}{\vdash A \to B \text{ type}} \;\to\; \text{form.}$$

- *Introduction rule*: Define the canonical elements of a type.

$$\frac{x : A \vdash b(x) : B}{\vdash x \mapsto b(x) : A \to B} \;\to\; \text{intro.}$$

- *Elimination rule*: Explains how to use a term in a derivation.

$$\frac{\vdash f : A \to B \qquad \vdash a : A}{\vdash f(a) : B} \;\to\; \text{elim.}$$

- *Computation rule*: Ensures compatibility between the introduction and elimination rules.

$$\frac{x : A \vdash b(x) : B, \qquad \vdash a : A}{\vdash (x \mapsto b(x))(a) = b(a)} \;\to\; \text{comp.}$$

Sometimes, a type can be defined as the "free" construction from a set of generators, with the generators being elements or functions with codomain the defined type. In this case, the set of rules defining the type can be inferred from the generators in a deterministic procedure. This kind of types are usually called *inductive types*. Let $A$ and $B$ be two types. Many of the most common types are in fact inductive types:

- The *sum type* $A + B$ is generated by the inclusion functions $\texttt{inl} : A \to A + B$ and $\texttt{inr} : B \to A + B$.
- The *product type* $A \times B$ is generated by all pairs $(\cdot\,,\cdot) : A \to (B \to A \times B)$.
- The *empty type* $\emptyset$ is the type freely generated without generators.
- The *unit type* $\mathbf{1}$ is generated by one element $* : \mathbf{1}$.
- The type of *natural numbers* $\mathbf{N}$ is generated by $0 : \mathbf{N}$ and a function $\texttt{succ} : \mathbf{N} \to \mathbf{N}$.

Let $B$ be a family indexed by $A$. We can generalize products and functions into the *dependent product* $\prod_{x:A} B(x)$, which has the same definition as the function type $A \to B$ but with the codomain type varying for each $x \in A$. The same can be done with sums, obtaining the *dependent sum* $\sum_{x:A} B(x)$, which is the type freely generated by the dependent function $(\,\cdot\,,\,\cdot\,) : \prod_{x:A}(B(x) \to A \times B(x))$.

Until now, we have seen how to construct types as "set-like" collections defined by rules. But in fact, type theory offers also a way to use types as logical propositions. When a type is interpreted as a proposition, it is considered to be true if it is inhabited, i.e., if it can be proven to have at least one term. This relation between type theory constructors and logical operators can be seen in Table 4.1. For more information about this relation, see [Uni13, Section 1.11].

| **Set theory** | **Type theory** |
|---|---|
| True | $\mathbf{1}$ |
| False | $\emptyset$ |
| Negation of $A$ | $A \to \emptyset$ |
| $A$ and $B$ | $A \times B$ |
| $A$ or $B$ | $A + B$ |
| $A$ implies $B$ | $A \to B$ |
| $A$ if and only if $B$ | $(A \to B) \times (B \to A)$ |
| $\forall\, x \in A,\, B(x)$ | $\prod_{x:A} B(x)$ |
| $\exists\, x \in A,\, B(x)$ | $\sum_{x:A} B(x)$ |

Table 4.1: Propositions as types

The more remarkable dependent type in Martin-Löf type theory is the *identity type*, denoted by $\mathsf{Id}_A(a,b)$ or $\mathsf{Id}(a,b)$, which serves as a logical equality inside of type theory. This type is also defined inductively, by a constructor $\mathsf{refl}_a : \mathsf{Id}_A(a,a)$ for every $a : A$. Observe that we can consider identity types of identity types, and so on recursively, which creates a higher dimensional structure for every type in the theory. An identity type of an identity type over a type $A$ will be called a 2-*identity type* over $A$. Following this procedure recursively defines $n$-*identity types*. Later in this chapter we will need to use the elimination and computation rules for identity types, which are usually called *path induction*. Then, following the procedure of [Uni13, Chapter 4], we can extract the elimination and computation rules from the generators:

$$\frac{\begin{array}{c} x : A,\, y : A,\, q : \mathsf{Id}_A(x,y) \vdash D(x,y,q) \text{ type} \\ \vdash p : \mathsf{Id}_A(a,b) \qquad x : A \vdash d(x) : D(x,x,\mathsf{refl}_x) \end{array}}{\vdash J_{A,D}(d,a,b,p) : D(a,b,p)} \ \mathsf{Id}\text{ elim.}$$

$$\frac{\begin{array}{c} x : A,\, y : A,\, q : \mathsf{Id}_A(x,y) \vdash D(x,y,q) \text{ type} \\ \vdash a : A \qquad x : A \vdash d(x) : D(x,x,\mathsf{refl}_x) \end{array}}{\vdash J_{A,D}(d,a,a,\mathsf{refl}_a) = d(a)} \ \mathsf{Id}\text{ comp.}$$

Using the identities, we can define a notion of equivalence between types. This notion of equivalence is similar to the definition of a homotopy equivalence, if we think of identities between functions as homotopies. We say that a function $f : A \to B$ is an *equivalence* if the following type is inhabited:

$$\mathsf{IsEquiv}(f) := \left( \sum_{g:A\to B} \mathsf{Id}(g \circ f, \mathsf{refl}_A) \right) \times \left( \sum_{h:A\to B} \mathsf{Id}(f \circ h, \mathsf{refl}_B) \right).$$

In fact, this property allows us to define the type of all equivalences between $A$ and $B$

$$A \simeq B := \sum_{f:A\to B} \mathsf{IsEquiv}(f).$$

Finally, *homotopy type theory (HoTT)* is defined as a Martin-Löf type theory with all the types previously introduced, which in addition has the univalence axiom and higher inductive types. The univalence axiom says that there is an equivalence between $\mathsf{Id}(A, B)$ and $A \simeq B$. This axiom is not very relevant for our further discussion; for more details see [Uni13]. On the other hand, higher inductive types extend the idea of inductive types, allowing us to use elements or functions on the identity types as generators. Thus, higher inductive types provide a mechanism for easily defining types with a higher structure.

In particular, we are interested in the type-theoretic versions of well-known topological spaces. The *type-theoretic circle* can be defined as the higher inductive type $\mathbf{S}^1$ generated as follows:

- A term $\mathsf{base} : \mathbf{S}^1$.
- A non-trivial identity $\mathsf{loop} : \mathsf{Id}_{\mathbf{S}^1}(\mathsf{base}, \mathsf{base})$.

Then, it can be understood as a type freely generated by a point and a non-trivial "loop" around that point. As in the case of inductive types, we could follow a deterministic procedure to extract all the rules governing the type $\mathbf{S}^1$. Similarly, the *type-theoretic torus* $\mathbf{T}^2$ is a higher inductive type with the following generators:

- A term $\mathsf{b} : \mathbf{T}^2$.
- Two different non-trivial identities named $\mathsf{p} : \mathsf{Id}_{\mathbf{T}^2}(\mathsf{b}, \mathsf{b})$ and $\mathsf{q} : \mathsf{Id}_{\mathbf{T}^2}(\mathsf{b}, \mathsf{b})$.
- A non-trivial 2-identity $\mathsf{t} : \mathsf{Id}_{\mathsf{Id}_{\mathbf{T}^2}(\mathsf{b},\mathsf{b})}(\mathsf{p} \cdot \mathsf{q}, \mathsf{q} \cdot \mathsf{p})$.

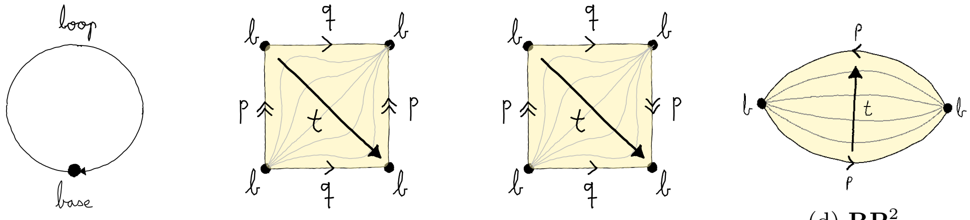
Inspired by the previous construction of the torus, the author in [Mar20] proposed defining the *type-theoretic Klein bottle* $\mathbf{K}$ as the higher inductive type with the following generators:

- A term $\mathsf{b} : \mathbf{K}$.
- Two different non-trivial identities named $\mathsf{p} : \mathsf{Id}_{\mathbf{K}}(\mathsf{b}, \mathsf{b})$ and $\mathsf{q} : \mathsf{Id}_{\mathbf{K}}(\mathsf{b}, \mathsf{b})$.
- A non-trivial 2-identity $\mathsf{t} : \mathsf{Id}_{\mathsf{Id}_{\mathbf{K}}(\mathsf{b},\mathsf{b})}(\mathsf{p} \cdot \mathsf{q}, \mathsf{q} \cdot \mathsf{p}^{-1})$.

Finally, following the proposal of Ripoll [Rip20], we define the *type-theoretic real projective plane* $\mathbf{RP}^2$ as the higher inductive type with the following generators:

- A term $\mathsf{b} : \mathbf{RP}^2$.
- A non-trivial identity $\mathsf{p} : \mathsf{Id}_{\mathbf{RP}^2}(\mathsf{b}, \mathsf{b})$.
- A non-trivial 2-identity $\mathsf{t} : \mathsf{Id}_{\mathsf{Id}_{\mathbf{RP}^2}(\mathsf{b},\mathsf{b})}(\mathsf{p}, \mathsf{p}^{-1})$.

The intuition behind the generators of these higher inductive types is to replicate the structure of the corresponding topological spaces as CW-complexes, with the structure of generators displayed visually in Figure 4.1.




(a) $\mathbf{S}^1$ (b) $\mathbf{T}^2$ (c) $\mathbf{K}$ (d) $\mathbf{RP}^2$

Figure 4.1: Diagrams of the generators of $\mathbf{S}^1$, $\mathbf{T}^2$, $\mathbf{K}$ and $\mathbf{RP}^2$.

Using the tools of homotopy type theory, several useful properties about these higher inductive types have been proven. Before considering the properties, we need to introduce two more types. Using higher inductive types, it is easy to define type-theoretic set quotients [Uni13, Section 6.10]. Then, we can define the *type-theoretic integers* $\mathbf{Z}$ as the usual quotient of $\mathbf{N} \times \mathbf{N}$ [Uni13, Remark 6.10.7]. Let $A$ be a type and $a : A$ be a term of $A$. Define the pointed type $(A, a)$ as a term of the dependent product $\prod_{A:\,\text{type}} A$. Then, define the *loop type* of $(A, a)$ as the type

$$\mathbf{\Omega}(A, a) := (\mathsf{Id}_A(a, a), \mathsf{refl}_a).$$

One of the first homotopical results shown in homotopy type theory was that the loop type of the circle is actually related to the integers, as in the case of the topological circle:

**Theorem 4.1.1.** [Uni13, Corollary 8.1.10]. *There is an equivalence* $\mathbf{\Omega}(\mathbf{S}^1, \mathsf{base}) \simeq \mathbf{Z}$.

The more complex relation between generators of the torus makes it difficult to obtain a result like the previous one directly. In 2016, Sojakova published an equivalence between the torus and the product of two circles. This equivalence enables us to easily transport results like the previous one from the circle to the torus:

**Theorem 4.1.2.** [Soj16]. *There is an equivalence* $\mathbf{T}^2 \simeq \mathbf{S}^1 \times \mathbf{S}^1$.

**Corollary 4.1.3.** *The type-theoretic loop type of the torus is*

$$\mathbf{\Omega}(\mathbf{T}^2, \mathsf{b}) \simeq \mathbf{Z} \times \mathbf{Z}.$$

*Proof.* Using Theorem 4.1.2, we know that an identity between elements of $\mathbf{T}^2$ can be identified with one of $\mathbf{S}^1 \times \mathbf{S}^1$. By [Uni13, Theorem 2.7.2], the identities between products are equivalent to products of identities. Then, the loop type must be equivalent to the product of the loop types of $\mathbf{S}^1$. Finally, by Theorem 4.1.1, this product must be equivalent to $\mathbf{Z} \times \mathbf{Z}$. $\square$

The case of the type-theoretic Klein bottle and the projective plane are even more complicated, and there is very little literature about them. In 2018, Hou and Harper [HH18] presented a formalization of covering spaces in homotopy type theory. Using this formalization, the author defined an alternative type-theoretic Klein bottle $\mathbf{K}'$ in [Mar20] as a type twofold covered by $\mathbf{T}^2$. In addition, the author proved that $\mathbf{K}'$ must be a 1-type, and conjectured that $\mathbf{K}'$ is equivalent to $\mathbf{K}$. If this conjecture could be proved, it would follow that $\mathbf{K}$ is indeed a 1-type, and that $\mathbf{\Omega}(\mathbf{T}^2, \mathsf{b})$ is a double cover of $\mathbf{\Omega}(\mathbf{K}, \mathsf{b})$. This last property implies that $\mathbf{\Omega}(\mathbf{K}, \mathsf{b})$ must be a group extension of index 2 of $\mathbf{Z} \times \mathbf{Z}$, by Corollary 4.1.3. Therefore, if proven, the conjecture that $\mathbf{K}' \simeq \mathbf{K}$ would give a useful description of the homotopical structure of the higher inductive type $\mathbf{K}$.

The $n$-th real projective space was defined in type theory by means of homotopy pushouts by Buchholtz and Rijke [BR17]. The formalization of homotopy pushouts in homotopy type theory can be found in [Uni13]. In [Rip20], the definition of the projective plane as a higher inductive type $\mathbf{RP}^2$ is conjectured to be equivalent to the definition of Buchholtz and Rijke. In addition, Ripoll presents strong evidence of this equivalence by describing the functions that could set up the equivalence.

## 4.2 Relation between types and $\infty$-groupoids

One of the main properties of Martin-Löf type theory is that it can be seen as a "syntax" for a certain class of categories. Those categories give the semantics to the theories developed inside type theory. We will call an *interpretation* any assignment of a category to a type theory, in such a way that all the rules and constructions are respected.

In the original works of Martin-Löf [Mar84], the identity types had different elimination rules, which "collapsed" the higher structure of identity types. Those kinds of type theories usually had semantics on the category of sets. Using the identity type presented in the previous section, Hofmann and Streicher [HS98] provided in 1998 the first interpretation with homotopical constructions based on the category of groupoids. The first full homotopical interpretation was due to Awodey and Warren [AW09], based on a particular kind of model categories. Later, van den Berg and Garner [BG12] made interpretations taking into account the coherence between the higher operations of an $\infty$-groupoid.

First, we need to review the interpretation of [AW09], which is the base of many of its successors. Let $\mathcal{C}$ be a cartesian closed model category. As we said before, we need to associate a categorical construction to each kind of judgement. Any judgement of type definition without context can be interpreted as a fibrant object of $\mathcal{C}$. A term declaration $\vdash a : A$ is interpreted as a global section to the fibrant object assigned to the judgement $\vdash A$ type. On the other hand, the equality judgements are interpreted as the identities of $\mathcal{C}$. Any dependent judgement is interpreted as a composition of fibrations, with as many fibrations as term declarations in the context.

The next step is constructing the basic types as defined in the previous section. The interpretation for each type must preserve the rules as properties in the category $\mathcal{C}$. For example, it can be proven that the sum type corresponds to the coproduct of $\mathcal{C}$, the product type to the product from $\mathcal{C}$, the unit type to the terminal object and the empty type to the initial one. Because $\mathcal{C}$ is a cartesian closed category, we can assign the function type to the exponential. The case of the dependent product and dependent sum is more complex. These constructions can be interpreted as a right adjoint and a left adjoint respectively to the pullback functor of the fibration of the dependent family. By arguments explained in detail in [AW09], the interpretation of the dependent sum always exists, but to ensure all the rules from the dependent product we need $\mathcal{C}$ to be locally cartesian closed. Finally, the identity type is interpreted as the path object. This interpretation has been summed up in Table 4.2, except for dependent sum and dependent product.

For example, in [AW09] it is shown that the category of simplicial sets has all the requirements mentioned earlier. By similar methods, in [BG12] it is shown that topological spaces also meet those requirements, but only if we take the Moore path space as path object. Then, we know that there is a simplicial set or a topological space related to each type.

Unfortunately, this interpretation method does not always give us clear information about which topological space is related to a given type. As mentioned earlier, the identity types endow a type with a higher structure. In fact, this structure resembles the structure

| Category theory | Type theory |
|---|---|
| Fibrant object $A$ | Type declaration $A$ type |
| Fibration $B \to A$ | Dependent family $x : A \vdash B(x)$ type |
| Initial object 0 | Empty type $\emptyset$ |
| Terminal object 1 | Unit type $\mathbf{1}$ |
| Global section $1 \to A$ | Term $x : A$ |
| Product $A \times B$ | Product $A \times B$ |
| Coproduct $A \sqcup B$ | Sum $A + B$ |
| Exponential object $A^B$ | Function $A \to B$ |
| Path object $\mathrm{Path}(A) \to A \times A$ | Identity type $a, b : A \vdash \mathsf{Id}_A(a, b)$ |

Table 4.2: Interpretation of type theory in a model category

expected in an $\infty$-groupoid [Uni13]: there is a tower of $n$-identities, with a weakly associative composition and a weak inverse. The presence of the tower of identity types follows directly from the definition. The following propositions imply the existence of composition, inverse and the expected properties:

**Proposition 4.2.1.** *Let $A$ be a type and $x, y : A$. Then, there exists a path inverse function*

$$\begin{array}{ccc} \mathsf{Id}_A(x,y) & \to & \mathsf{Id}_A(y,x) \\ p & \mapsto & p^{-1} \end{array}$$

*such that* $\mathsf{refl}_x{}^{-1} = \mathsf{refl}_x$ *for all* $x : A$.

*Proof.* For all $x, y : A$ and $p : \mathsf{Id}_A(x,y)$ we want to define $p^{-1} : \mathsf{Id}_A(y,x)$. By path induction, it is enough to consider the case of $x = y$ and $p = \mathsf{refl}_x$. In this case, we want to define $\mathsf{refl}_x{}^{-1} : \mathsf{Id}_A(x,x)$, and then we can simply take $\mathsf{refl}_x{}^{-1} := \mathsf{refl}_x$. □

**Proposition 4.2.2.** *Let $A$ be a type and $x, y, z : A$. Then, there exists a composition function*

$$\begin{array}{ccc} \mathsf{Id}_A(x,y) \times \mathsf{Id}_A(y,z) & \to & \mathsf{Id}_A(x,z) \\ (p,q) & \mapsto & p \cdot q \end{array}$$

*such that* $\mathsf{refl}_x \cdot \mathsf{refl}_x = \mathsf{refl}_x$ *for all* $x : A$.

*Proof.* By the elimination rule of the product, presented in [Uni13, p. 38], any element $\mathsf{Id}_A(x,y) \times \mathsf{Id}_A(y,z) \to \mathsf{Id}_A(x,z)$ can be defined as a function

$$\mathsf{Id}_A(x,y) \to (\mathsf{Id}_A(y,z) \to \mathsf{Id}_A(x,z)).$$

To define this function, we need to apply path induction twice: one for the first parameter, and another for the second one. This double path induction is equivalent to assuming $x = y = z$ and $p = q = \mathsf{refl}_x$. Then, we can define $\mathsf{refl}_x \cdot \mathsf{refl}_x := \mathsf{refl}_x : \mathsf{Id}_A(x,x)$. □

**Proposition 4.2.3.** *Let $A$ be a type, $a, b, c, d : A$, $p : \mathsf{Id}_A(a,b)$, $q : \mathsf{Id}_A(b,c)$ and $r : \mathsf{Id}_A(c,d)$. Then the following types are inhabited:*

(i) $\mathsf{Id}(p, p \cdot \mathsf{refl}_b)$ *and* $\mathsf{Id}(p, \mathsf{refl}_a \cdot p)$.

(ii) $\mathsf{Id}(p^{-1} \cdot p, \mathsf{refl}_b)$ *and* $\mathsf{Id}(p \cdot p^{-1}, \mathsf{refl}_a)$.

(iii) $\mathsf{Id}((p^{-1})^{-1}, p)$.

(iv) $\mathsf{Id}(p \cdot (q \cdot r), (p \cdot q) \cdot r)$.

*Proof.* By Proposition 4.2.1 and Proposition 4.2.2, we know that there are two judgmental equalities $(a)$ $\mathsf{refl}_x{}^{-1} = \mathsf{refl}_x$ and $(b)$ $\mathsf{refl}_a = \mathsf{refl}_a \cdot \mathsf{refl}_a$.

(i) By path induction on $p$, it is enough to prove that there is an inhabitant in the case where $a = b$ and $p = \mathsf{refl}_a$. Then, the equality $(a)$ implies that there is a canonical element

$$\mathsf{refl}_{\mathsf{Id}(\mathsf{refl}_a, \mathsf{refl}_a)} : \mathsf{Id}(\mathsf{refl}_a, \mathsf{refl}_a) = \mathsf{Id}(\mathsf{refl}_a, \mathsf{refl}_a \cdot \mathsf{refl}_a).$$

The same argument proves the existence of a term $\mathsf{Id}(p, \mathsf{refl}_a \cdot p)$.

(ii) By path induction on $p$, consider $a = b$ and $p = \mathsf{refl}_a$. Then, the following chain of equalities follows from the equations $(a)$ and $(b)$, and proves the existence of a canonical term $\mathsf{refl}_a{}^{-1} \cdot \mathsf{refl}_a = \mathsf{refl}_a \cdot \mathsf{refl}_a = \mathsf{refl}_a$.

(iii) As in the previous cases, use path induction on $p$ with $a = b$ and $p = \mathsf{refl}_a$. Then, using the equation $(a)$, a canonical term exists because $(\mathsf{refl}_a{}^{-1})^{-1} = \mathsf{refl}_a{}^{-1} = \mathsf{refl}_a$.

(iv) Use path induction on $p$, $q$ and $r$. Then, using the equation $(b)$, a canonical term exists because

$$\mathsf{refl}_a \cdot (\mathsf{refl}_a \cdot \mathsf{refl}_a) = \mathsf{refl}_a \cdot \mathsf{refl}_a = (\mathsf{refl}_a \cdot \mathsf{refl}_a) \cdot \mathsf{refl}_a.$$ □

The study of the structure generated by the tower of identity types led to a different family of models of homotopy type theory. The original suggestion of Grothendieck [Gro83] for a model of $\infty$-groupoids was to use globular sets. A *globular set* is a diagram of sets and functions

$$X_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} X_1 \underset{t_1}{\overset{s_1}{\leftleftarrows}} X_2 \underset{t_2}{\overset{s_2}{\leftleftarrows}} \cdots$$

satisfying the *globular identities*

$$s_n \circ s_{n+1} = s_n \circ t_{n+1},$$
$$t_n \circ s_{n+1} = t_n \circ t_{n+1}.$$

The category of globular sets is denoted by **gSet**. Furthermore, define a *globular object* on a category $\mathcal{C}$ as a diagram with the shape of a globular set but with objects and morphisms of $\mathcal{C}$ satisfying the globular identities. To define the model of $\infty$-groupoids as globular sets we need to add some structure over globular sets, following the construction of Batanin [Bat98]. A *strict $\infty$-category* is defined as a globular set $\mathcal{C}$. Each set $\mathcal{C}_n$ represents the set of $n$-morphisms and the source and target correspond to the domain and codomain $(n-1)$-morphisms. Additionally, we need to ask for the existence of a composition of $n$-morphisms along a common boundary in any lower dimension, satisfying associativity, unit and interchange laws; see [Ber02] for details.

Thus, every strict $\infty$-category has an underlying globular set, which defines a forgetful functor $U$ and a monadic adjunction with a free functor $F$ between the category of globular sets and the one from strict $\infty$-categories. This monadic adjunction yields a *free strict $\infty$-category* monad $(T, \mu, \eta)$ on **gSet**. In particular, $T1$ (where 1 denotes the terminal globular set, with just one element of each dimension) consists informally of free "pasting" elements of 1, including degenerate pastings from the identity elements of $F1$.

A *globular operad* is a monad $P$ on **gSet** equipped with a cartesian monad morphism $\rho : P \Rightarrow T$. Additionally, a $P$-*algebra* is a globular set $A$ together with an action of $P$ on $A$, i.e., a composition map $c : TA \times_{T1} P \to A$ satisfying some technical conditions. One needs to ask two additional properties on globular operads. First, a globular operad is *normalized* if there is a natural bijection $(PX)_0 \cong X_0$. Second, a globular operad is *contractible* if it is a "deformation" of the monad $T$. This fact is expressed by some technical conditions found in [Ber02].

Following the ideas of Grothendieck, several authors presented models of $\infty$-categories and $\infty$-groupoids based on globular sets. The definition presented here follows the work of Batanin [Bat98], and the survey done by Berger [Ber02]. A *Batanin $\infty$-category* $(P, X)$ is an algebra $X$ for a contractible normalized globular operad $P$. For any Batanin $\infty$-category $(P, X)$, an equivalence $x \simeq y$ between two $n$-morphisms is given by two $(n+1)$-morphisms $f : x \to g$ and $g : y \to x$ such that there exist equivalences $g \circ f \simeq \mathrm{Id}_x$ and $f \circ g \simeq \mathrm{Id}_y$. Then, an $(n+1)$-morphism $f : x \to y$ is weakly invertible if it is part of an equivalence $x \simeq y$. Finally, a *Batanin $\infty$-groupoid* is a Batanin $\infty$-category with every element of any dimension being weakly invertible with respect to every system of compositions on $P$. The homotopy hypothesis for this particular model is proven in the work of Batanin [Bat98], using and adjunction between topological spaces and the category of Batanin $\infty$-groupoids which induces an equivalence at the homotopy categories.

Observe that the tower of identity types in homotopy type theory has an inherent globular structure. The edges of an identity define the source and target maps, and the globular identities follow directly. On the other hand, the weak composition and weak unit coincide with the ones found in the monadic structure of the normalized contractible globular operads. Finally, the weak inverses from type theory have the same definition as the ones from a Batanin $\infty$-category, and they exist for every element of an identity, as in the case of a Batanin $\infty$-groupoid.

The first appearance of this idea was in the thesis of Warren [War08]. Later, it was further developed in the independent works of van der Berg and Garner [BG10], and Lumsdaine [Lum09]. In [BG10], the authors build a category from the type theory, by similar methods to the interpretations presented earlier, and then show that types in that category are algebras over a normalized contractible globular operad with weakly invertible morphisms.

## 4.3 Modeling types as Moore path categories

In this section we want to assess whether the model of the fundamental $\infty$-groupoid as a Moore path category can help to interpret results from type theory. Our first try was to make an interpretation directly onto the category of $\infty$-groupoids as topological categories, following the methods of [AW09]. But this approach cannot succeed because the corresponding model of $\infty$-**Grpd** fails to be cartesian closed and locally cartesian closed [Rie19].

In the rest of this work, we will use the interpretation of type theory presented in [BG12]. This article presents a mix of the two previous methods: it interprets type theory in several model categories, but at the same time it chooses path objects in a way that they present the structure of a normalized contractible globular operad with weakly invertible morphisms. In particular, we are interested in the example of interpreting onto the category of topological spaces, choosing as path objects the Moore path spaces.

For any type $A$, we denote by $\mathcal{X}(A)$ the topological space which interprets $A$ following the work of [BG12]. For each term $a : A$, we know that there is an element of $\mathcal{X}(A)$ which will be denoted $\bar{a} \in \mathcal{X}(A)$. Then, for each $x, y : A$, the identity type $\mathsf{Id}_A(x, y)$ will be interpreted as $\mathcal{X}(\mathsf{Id}_A(x, y)) = P^M_{\bar{x},\bar{y}}\mathcal{X}(A)$.

Then, we can use Moore path categories as a tool to realize the globular structure of the tower of identities. Each term of a type $A$ has an associated element of $\mathcal{X}(A)$, which itself corresponds to an object of $\Pi^M_\infty(\mathcal{X}(A))$. Additionally, for every two terms $x, y : A$, there is a homset which must be equal to

$$\Pi^M_\infty(\mathcal{X}(A))(\bar{x}, \bar{y}) = P^M_{\bar{x},\bar{y}}\mathcal{X}(A) = \mathcal{X}(\mathsf{Id}_A(x, y)).$$

The composition and weak inverses from $\Pi^M_\infty(\mathcal{X}(A))$ realize the ones found in type theory. The 2-identities will be modelled as morphisms of $\Pi^M_\infty(\mathcal{X}(\mathsf{Id}_A(x, y))) = \Pi^M_\infty(P^M_{\bar{x},\bar{y}}\mathcal{X}(A))$ for every $x, y : A$, and so on recursively. In this case the associativity of the Moore path category is stricter than the one in type theory, but this also happens in other interpretations like the model in strict $\infty$-groupoids of [War08]. In fact, this identification also realizes the globular structure of the identity types, as shown in [BG12, Proposition 5.1.1].

| **Level** | **$\infty$-groupoids** | **Types** |
|---|---|---|
| 0 | $\Pi^M_\infty(\mathcal{X}(A))$ | $A$ |
| 1 | $\coprod_{\bar{x},\bar{y}\in\mathcal{X}(A)} \Pi^M_\infty(P^M_{\bar{x},\bar{y}}\mathcal{X}(A))$ | $\sum_{x,y:A} \mathsf{Id}(x, y)$ |
| 2 | $\coprod_{\bar{x},\bar{y}\in\mathcal{X}(A)} \coprod_{\bar{p}_1,\bar{q}_1\in P^M_{\bar{x},\bar{y}}\mathcal{X}(A)} \Pi^M_\infty(P^M_{\bar{p}_1,\bar{q}_1}(P^M_{\bar{x},\bar{y}}\mathcal{X}(A)))$ | $\sum_{x,y:A} \sum_{p_1,q_1:\mathsf{Id}(x,y)} \mathsf{Id}(p_1, q_1)$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

Table 4.3: Identification of the globular structure of Moore path categories and types

As an example, we will use Moore path categories to ensure that the topological space related to some higher inductive types is the expected one. Our targets will be the circle and the torus defined as higher inductive types. In type theory, $\mathbf{Z}$ is a set, which implies that it does not have any non-trivial higher identities. Then, the $\infty$-groupoid of $\Pi^M_\infty(\mathcal{X}(\mathbf{Z}))$ must have as objects the set $\mathbb{Z}$ and trivial homsets. Therefore, $\mathcal{X}(\mathbf{Z})$ must be equivalent to $\mathbb{Z}$, because the two realize the same fundamental $\infty$-groupoid.

**Proposition 4.3.1.** *The topological spaces $\mathcal{X}(\mathbf{S}^1)$ and $\mathbb{S}^1$ have the same homotopy type.*

*Proof.* By Theorem 2.4.8, we know that $\Pi^M_\infty(\mathbb{S}^1)$ is homotopy equivalent to the fundamental $\infty$-groupoid of $\mathbb{S}^1$. Because $\mathbb{S}^1$ is path connected, we can choose any basepoint $x \in \mathbb{S}^1$, and we have a homotopy equivalence $\Pi^M_\infty(\mathbb{S}^1) \simeq \mathbb{D}(\Omega^M_x\mathbb{S}^1)$. Then, $\Pi^M_\infty(\mathbb{S}^1)$ is homotopy equivalent to an $\infty$-groupoid with one object $x$ and the homset $\Omega^M_x\mathbb{S}^1$. On the other hand, we know that the circle has $\Omega_x\mathbb{S}^1 \simeq \mathbb{Z}$. By Proposition 2.4.2, we also have $\Omega^M_x\mathbb{S}^1 \simeq \Omega_x\mathbb{S}^1 \simeq \mathbb{Z}$. The existence of this weak equivalence implies that the fundamental $\infty$-groupoid of $\mathbb{S}^1$ is weakly homotopy equivalent to an $\infty$-groupoid with one object and $\mathbb{Z}$ as homset.

Because of Theorem 4.1.1, the identity structure over $\mathbf{S}^1$ must have the only identity type equivalent to $\mathbb{Z}$, and all higher identities being trivial. Then, we know that the topological space $\mathcal{X}(\mathbf{S}^1)$ associated to the circle $\mathbf{S}^1$ has to have $\Pi^M_\infty(\mathcal{X}(\mathbf{S}^1))$ weakly equivalent to one with only one object $x$ and the only non-trivial homset $\mathbb{Z}$. But this coincides with the fundamental $\infty$-groupoid of the circle found earlier. Therefore, $\mathcal{X}(\mathbf{S}^1)$ must be homotopy equivalent to the topological circle $\mathbb{S}^1$. □

**Proposition 4.3.2.** *The topological spaces $\mathcal{X}(\mathbf{T}^2)$ and $\mathbb{T}^2$ have the same homotopy type.*

*Proof.* The fundamental $\infty$-groupoid of $\mathbb{T}^2$ is homotopy equivalent to an $\infty$-groupoid with one object $x$ and the homset $\Omega_x^M \mathbb{T}^2$. Furthermore, the torus is the classifying space of $\mathbb{Z} \times \mathbb{Z}$. Hence, $\mathbb{Z} \times \mathbb{Z} \simeq \Omega_x \, \mathrm{B}(\mathbb{Z} \times \mathbb{Z}) \cong \Omega_x \mathbb{T}^2 \simeq \Omega_x^M \mathbb{T}^2$ for any base point $x \in \mathbb{T}^2$. Thus, the fundamental $\infty$-groupoid of $\mathbb{T}^2$ is weakly homotopy equivalent to an $\infty$-groupoid with one object and $\mathbb{Z} \times \mathbb{Z}$ as homset. Thanks to Corollary 4.1.3, the same argument used in the previous proposition proves that the torus $\mathbb{T}^2$ has the same homotopy type as $\mathcal{X}(\mathbf{T}^2)$. □

To reproduce the same procedure for the Klein bottle, first we would need to prove the conjecture that the type-theoretical torus is a double cover of the type-theoretical Klein bottle, which would imply that the latter is a 1-type. Furthermore, in the case of 1-types defined as higher inductive types, there is work by Veltri and van der Weide [VW20] providing a much more general interpretation.

As we stated in the Introduction, we are interested in a way to define syntactically a type-theoretical version of a finite CW-complex space as a higher inductive type. In addition, we want that the interpretation of that higher inductive type on any category of $\infty$-groupoids has the same homotopy type as the fundamental $\infty$-groupoid of the original space. Although we have used Moore path categories as intended for simple examples like the circle or the torus, further research is needed for studying other cases like the real projective spaces with other techniques.

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