Limit sketches and presentability

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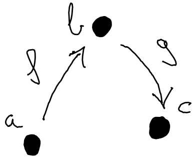
Categories



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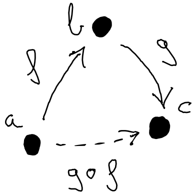

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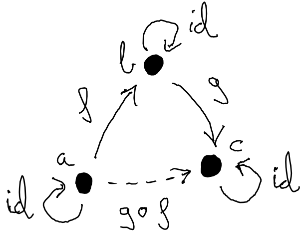

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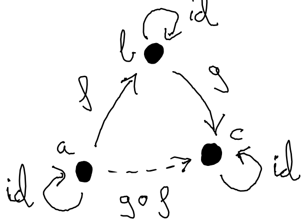
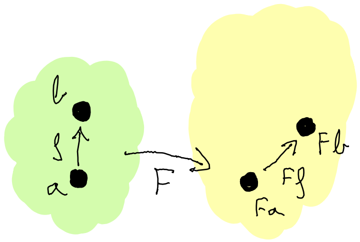


Functors and natural transformations



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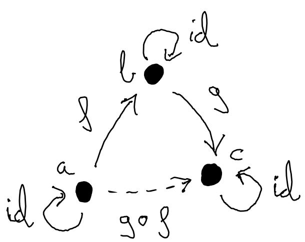

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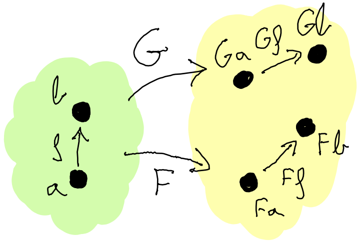

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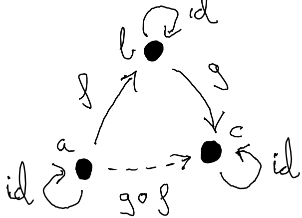

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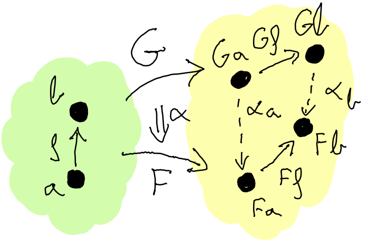

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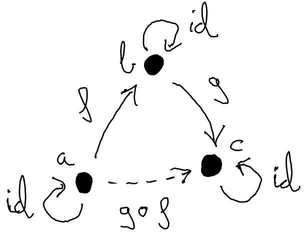

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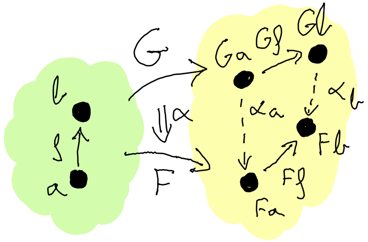

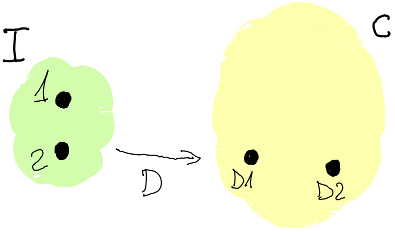

Cones and limits

- Diagram $D : I \to C$

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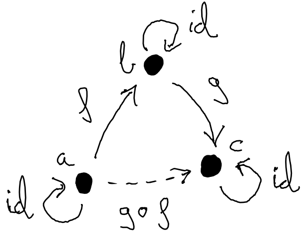

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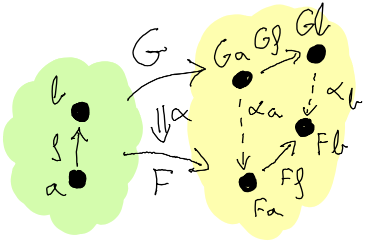

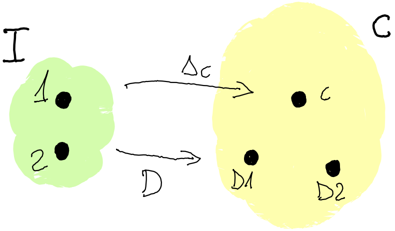

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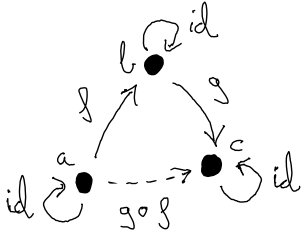

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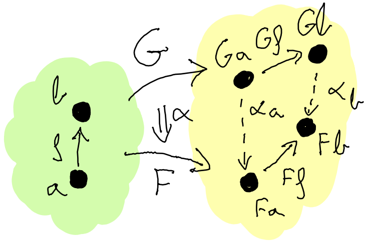

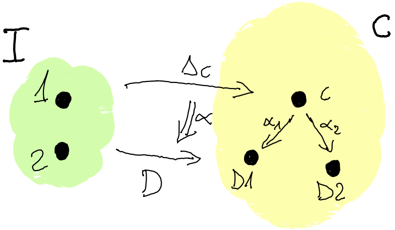

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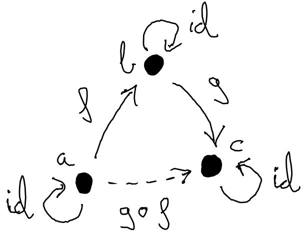

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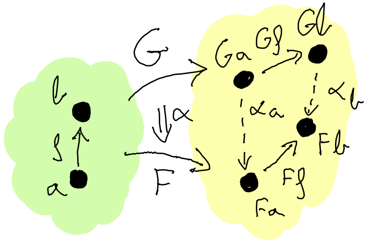

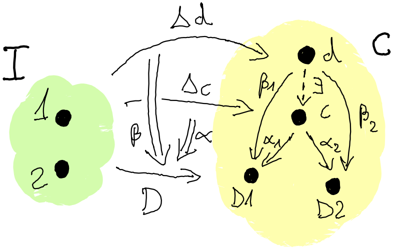

Cones and limits

- Diagram $D : I \to C$
- Cone $\alpha : \Delta c \Rightarrow D$
- α is a limit if for all $d \in C$

$$\mathrm{Hom}(d, c) \cong \mathrm{Cones}(D, d)$$

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Examples. Set, Grp, sSet, ... **Non example.** Top

Limit sketches

A **limit sketch** (Bastiani and Ehresmann 1972) is a pair (A, C) of a small category A and a set of cones C over A.

Example. Let A be the small category generated by the square (a).

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & \text{(a)} & \downarrow \\ 2 & \longrightarrow & 3 \end{array}$$

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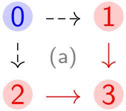

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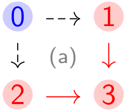

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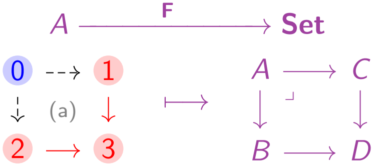

A model **F** of the sketch $(A, \{c\})$ is a pullback of sets 👍

Representation theorem

Theorem (Adamek and Rosicky 1994)

The following are equivalent:

(i) *Presentable categories.*

(ii) *Limit-sketchable categories.*

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Goal

Presentable ∞-categories $\overset{?}{\simeq}$ Limit-sketchable ∞-categories

Plan

Presentable ∞-categories

Limit ∞-sketches

Representation theorem

Plan

Presentable ∞-categories

Limit ∞-sketches

Representation theorem

Informal higher categories

A **higher category** has objects and:

- ☑ n-morphisms between $(n-1)$-morphisms for all $n \geq 1$,
- ☑ Composition, identities and associativity of n-morphisms weakly up to a $(n+1)$-morphism for all $n \geq 1$.

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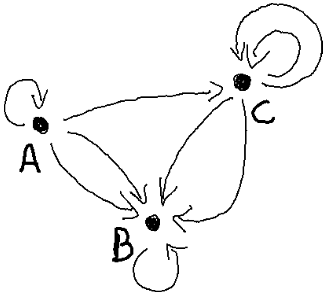

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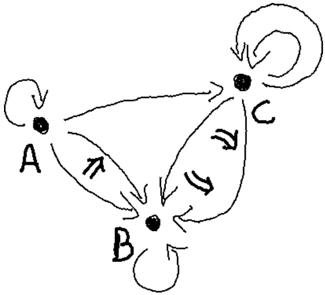

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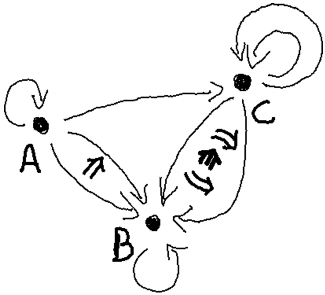

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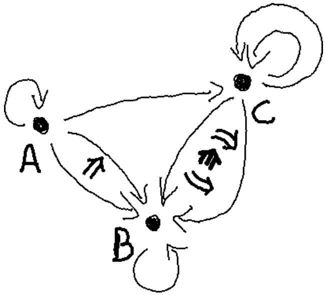

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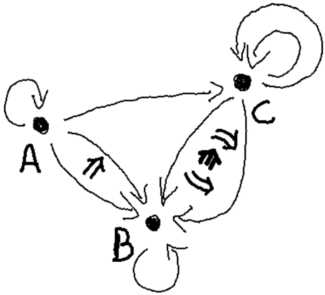

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- ∞-**category** $:= (\infty, 1)$-category
- ∞-**groupoid** $:= (\infty, 0)$-category

Limits and colimits

Let $\mathcal{C}$ be an ∞-category, and I be a small ∞-category. Given any object $x \in \mathrm{Obj}(\mathcal{C})$, the **constant diagram** $\Delta x : I \to \mathcal{C}$ sends all objects of I to x, and all higher morphisms to higher identities.

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Cocones and **colimit cocones** are defined as cones and limit cones in the opposite ∞-category.

Accessibility

Let κ denote a regular cardinal and $\mathcal{C}$ an ∞-category.

- An ∞-category $\mathcal{K}$ is κ-**filtered** if, for every κ-small ∞-category I, every diagram $D : I \to \mathcal{K}$ admits a cocone $\alpha : D \Rightarrow \Delta x$.

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An ∞-category $\mathcal{C}$ is **accessible** if it is locally small and there is a regular cardinal κ such that:

- ☑ $\mathcal{C}$ admits κ-filtered colimits.
- ☑ There is some essentially small sub-∞-category of κ-compact objects which generates $\mathcal{C}$ under κ-filtered colimits.

Presentability

Definition

An ∞-category is **presentable** if it is accessible and cocomplete.

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Example

(a) The ∞-category of homotopy types $\mathcal{S}$ is presentable.

(b) Any ∞-topos is presentable.

(c) The nerve of any presentable 1-category is presentable.

(d) If $\mathcal{A}$ is a small ∞-category and $\mathcal{C}$ is a presentable ∞-category, then $\mathrm{Fun}(\mathcal{A}, \mathcal{C})$ is presentable.

Plan

Presentable ∞-categories

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A **limit ∞-sketch** (Joyal 2008) $\mathcal{T} = (\mathcal{K}, \mathfrak{L})$ is a small ∞-category $\mathcal{K}$ equip with a set $\mathfrak{L}$ of cones.

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Let $\mathcal{C}$ be a complete ∞-category. A functor $F : \mathcal{K} \to \mathcal{C}$ is a **model** of a limit ∞-sketch $\mathcal{T} = (\mathcal{K}, \mathfrak{L})$ in $\mathcal{C}$ if it sends each cone in $\mathfrak{L}$ to a limit cone in $\mathcal{C}$.

$$\mathrm{Mod}(\mathcal{T}, \mathcal{C}) \coloneqq \infty\text{-category of models of } \mathcal{T} \text{ in } \mathcal{C}$$
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We say that an ∞-category is **limit ∞-sketchable** (or **essentially ∞-algebraic**) if it is equivalent to the ∞-category of models of some limit ∞-sketch.

Examples: ∞-algebraic theories

An ∞-**algebraic theory** (or ∞-**Lawvere theory**) is a small ∞-category with finite products. A **model** (or **algebra**) for an ∞-algebraic theory $\mathcal{A}$ is a functor $\mathcal{A} \to \mathcal{S}$ that preserves products.

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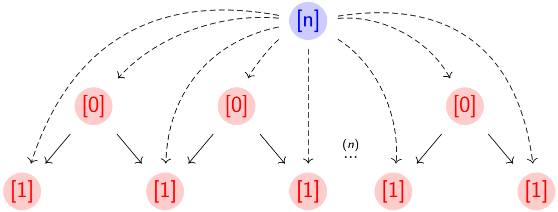
Theorem (Rosicky 2007 and Lurie 2009)

The ∞-category of models of an ∞-algebraic theory is presentable.

Examples: Internal precategories

Let $\mathcal{C}$ be a complete ∞-category, $\mathcal{A}$ be the nerve of Δ^{op}, and c_n be the cone with apex and diagram for all $n \in \mathbb{N}$:



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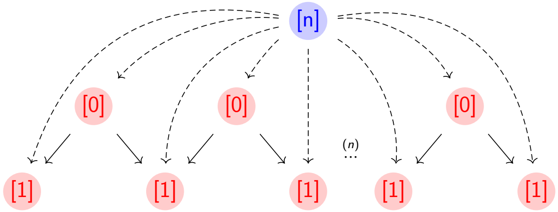
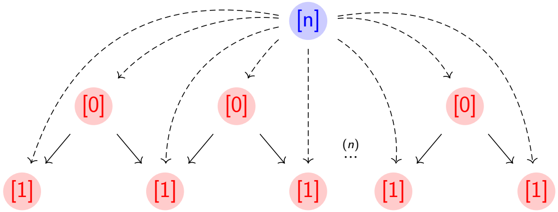


Then $\mathcal{T} = (\mathcal{A}, \{c_n \mid n \in \mathbb{N}\})$ is a limit ∞-sketch, and a model $F : \mathcal{A} \to \mathcal{C}$ is a simplicial object in $\mathcal{C}$ such that

$$F_n \xrightarrow{\sim} F_1 \times_{F_0} F_1 \times_{F_0} \cdots \times_{F_0} F_1. \qquad \textit{(Segal condition)}$$

Examples: Internal precategories

Let $\mathcal{C}$ be a complete ∞-category, $\mathcal{A}$ be the nerve of Δ^{op}, and c_n be the cone with apex and diagram for all $n \in \mathbb{N}$:



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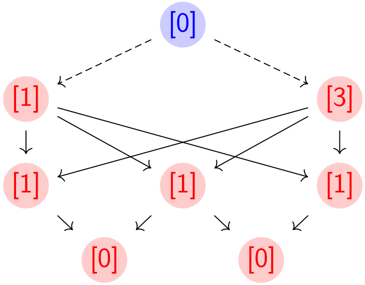
$\mathrm{Mod}(\mathcal{T}, \mathcal{C}) \simeq$ **Internal precategories**

$\mathrm{Mod}(\mathcal{T}) \simeq$ **Segal spaces**

Examples: Internal univalent categories

Let $\mathcal{A}$ be as before, $\mathfrak{L}_S$ be the set of cones of the previous sketch, and d be the cone with apex and diagram:



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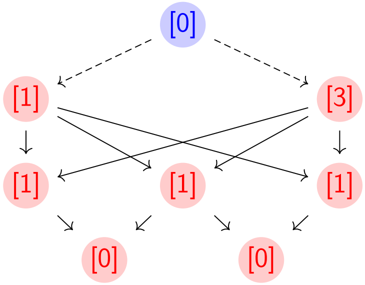

Then $\mathcal{T} = (\mathcal{A}, \mathfrak{L}_S \cup \{d\})$ is a limit ∞-sketch, and a model $F : \mathcal{A} \to \mathcal{C}$ is an internal precategory in $\mathcal{C}$ such that

$$\begin{array}{ccc} F_0 & \longrightarrow & F_3 \\ \big\downarrow & \lrcorner & \big\downarrow \\ F_1 & \longrightarrow & F_1 \times_{F_0}^{d_1, d_1} F_1 \times_{F_0}^{d_0, d_0} F_1 \end{array}$$

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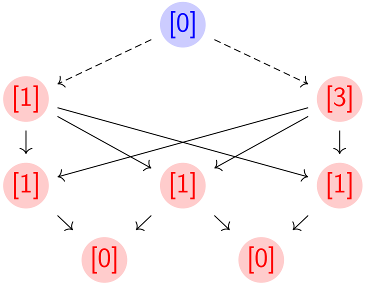

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$$\mathrm{Mod}(\mathcal{T}, \mathcal{C}) \simeq \textbf{Internal univalent categories}$$

$$\mathrm{Mod}(\mathcal{T}) \simeq \textbf{Complete Segal spaces}$$

Plan

Presentable ∞-categories

Limit ∞-sketches

Representation theorem

Representation theorem

Theorem (M.)

An ∞-category is presentable $\iff$ it is limit ∞-sketchable.

Corollary

The ∞-category of models of a limit ∞-sketch in a presentable ∞-category is presentable.

Future work

- **Generalization:** A ∞-category is accessible if, and only if, it is equivalent to the ∞-category of models of an ∞-sketch.

  A **sketch** is a limit sketch with a set of cocones which are sent to colimit cocones by any model.

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  $\Longrightarrow$ **Model-independent** version of this presentation!

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  $\implies$ **Model-independent** version of this presentation!

- Formalize this work with a proof assistant which supports synthetic ∞-categories like `rzk`.

References

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Thank you for listening!

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Limit sketches and presentability

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