Grothendieck's homotopy hypothesis

David Martínez Carpena

October 25, 2021

Abstract

In higher category theory, ∞-groupoids are ∞-categories whose morphisms are weakly invertible at all orders. Every topological space has an associated ∞-groupoid, named its fundamental ∞-groupoid, which encodes the information of higher paths over the space. The statement that every space can be recovered up to homotopy from its fundamental ∞-groupoid is known as Grothendieck's homotopy hypothesis. In this talk we present the model of ∞-categories based on topologically enriched categories, and discuss the homotopy hypothesis in this context.

1 Introduction

Notation: An ∞-category is a $(\infty, 1)$-category, an ∞-grupoid is a $(\infty, 0)$-category, and **Top** is the category of weakly Hausdorff compactly generated spaces.

The first occurrence of a general homotopy hypothesis was in a letter from Grothendieck to Quillen, which was published as an introduction to Grothendieck's "Pursuing Stacks" [Gro83]. The goal of this book is to study homotopy theory via category theory, higher types and stacks. The original statement of the homotopy hypothesis from Grothendieck is the following:

> Any homotopy type is "essentially the same" as an ∞-grupoid up to ∞-equivalence.
> (On page 5 of [Gro83])

If we choose a model of ∞-grupoids, the homotopy hypothesis applied to this model can be from a tautology to a non-trivial theorem. Following the techniques from [Lur09], we can show that (combinatorial simplicial) model categories encode the ∞-category of ∞-grupoids. Then, the "essentially the same" part of the homotopy hypothesis is equivalent to showing a zigzag of Quillen equivalences between the model structures of topological spaces and ∞-grupoids.

The goal of this talk is to prove the homotopy hypothesis for **tCat** (which is equivalent to **sCat** as seen in the previous talk) with a zigzag of Quillen equivalences from **Top** to a model structure over the ∞-grupoids in the **tCat** model.

2 Model of ∞-categories as topological categories

As introduced in the previous talk, **tCat** denotes the category of all topological categories with the Bergner model structure [Ber08; Amr13]. It can be seen that each topological category models an ∞-category: the n-morphisms are defined as $(n-1)$-homotopies in the spaces of morphisms. The properties of topological homotopies (weak associativity, unit and invertibility) hold for all $n > 1$, and we have strict associativity and unit but not invertibility for 1-morphisms. Then, the following definition is natural

Definition 2.1. A topological category $\mathcal{C}$ is an *∞-groupoid* if $\pi_0\mathcal{C}$ is a groupoid. The subcategory of ∞-groupoids will be denoted ∞-**Grpd**.

Recall that for any cocomplete category $\mathcal{C}$ and small category S and any functor $F : S \to \mathcal{C}$, the *formal nerve* N_F and the *formal realization* $|\cdot|_F$ are the pair of adjoint functors $(|\cdot|_F, \mathrm{N}_F)$ defined by the following commutative diagram with Y being the Yoneda embedding:

$$\begin{tikzcd}
S \arrow[r, "F"] \arrow[d, "Y"'] & \mathcal{C} \arrow[dl, shift right, "\mathrm{N}_F"'] \\
\mathbf{Set}^{S^{op}} \arrow[ur, shift right, "|\cdot|_F"'] &
\end{tikzcd}$$

The relation between **Top** to ∞-**Grpd** arises from a chain of adjunctions between those categories, where each adjunction is defined as a formal nerve and realization or an enriched version of one.

$$\begin{tikzcd}
\mathbf{Top} \arrow[rr, bend right, "\mathrm{Sing}"'] & ① & \mathbf{sSet}_Q \arrow[ll, bend right, "|\cdot|"'] \arrow[rr, bend left, "k_!"] & ④ & \mathbf{sSet}_J \arrow[ll, bend left, "k^!"] \arrow[rr, bend left, "\mathfrak{C}"] & ③ & \mathbf{sCat} \arrow[ll, bend left, "\mathrm{N}_{hc}"] \arrow[rr, bend left, "|\cdot|"] & ② & \mathbf{tCat} \arrow[ll, bend left, "\mathrm{Sing}"]
\end{tikzcd}$$

① The singular simplicial set Sing and the geometric realization $|\cdot|$ are defined as formal nerve and realization by the cosimplicial object $\Delta^\bullet : \Delta \to \mathbf{Top}$ where Δ^n is the standard topological simplex. In addition, it is well-known that those functors form a Quillen equivalence.

② As seen in the previous talk, any Quillen equivalence can be lifted to an enriched one. This is done with the previous Quillen equivalence between **Top** and **sSet**.

③ The homotopy coherent nerve N_{hc} and the simplicial path $\mathfrak{C}$ are defined as formal nerve and realization by the cosimplicial object $\Delta^{hc} : \Delta \to \mathbf{sCat}$ which sends any $[n] \in \Delta$ to the simplicial category $(FU)[n]$ as defined in the previous talk [Rie11]. Equivalently, Δ^{hc} sends any $[n] \in \Delta$ to the simplicial category with $\{0, 1, \ldots, n\}$ as set of objects and morphisms:

$$\Delta^{hc}[n](i,j) = \begin{cases} \emptyset & j < i \\ * & i = j \\ (\Delta[1])^{(j-i-1)} & j > i \end{cases}$$

As proven by Lurie [Lur09], $(\mathfrak{C}, \mathrm{N}_{hc})$ form a Quillen equivalence.

④ The functors $k^!$ and $k_!$ are defined as formal nerve and realization by the cosimplicial object $k : \Delta \to \mathbf{sSet}$ which sends any $[n] \in \Delta$ to the nerve of the groupoid freely generated by $[n]$ as a category. As shown in [JT07, Theorem 1.19], $(k_!, k^!)$ form a Quillen adjunction betwen $\mathbf{sSet}_Q$ and $\mathbf{sSet}_J$, but not a Quillen equivalence.

In fact, $(k_!, k^!)$ can be seen as a localization adjunction between quasi-categories and Kan complexes:

Proposition 2.2. [JT07, Proposition 1.16 and 1.20]. *There is a functor J from quasi-categories to Kan complexes, defined as $J(X)$ being the largest sub-Kan complex of a quasi-category X. In addition, the following are true:*

(i) *The natural map $k^!(X) \to J(X)$ is a trivial fibration for every quasi-category X.*

(ii) *The natural map $X \to k_!(X)$ is a monic weak equivalence for every simplicial set X.*

Then, $k_!$ is weak equivalent to the inclusion of Kan complexes in quasi-categories and $k^!$ is weak equivalent to J, thus sending quasi-categories to their largest sub-Kan complex.

Theorem 2.3. [JT07, Proposition 1.15]. *The Quillen model structure on* **sSet** *is a left Bousfield localization of the Joyal model structure on* **sSet**. *Thus, every Joyal's weak equivalence is a Quillen's weak equivalence. Furthermore, a map between Kan complexes is a Quillen's weak equivalence if and only if it is a Joyal's weak equivalence.*

3 The homotopy hypothesis for topological categories

As previously stated, our final goal is to prove the homotopy hypothesis for topological categories. Let $\psi = k^! \circ \mathrm{N}_{hc} \circ \mathrm{Sing}$ and $\theta = |\cdot| \circ \mathfrak{C} \circ k_!$. It can be seen that these functors restrict to a Quillen equivalence between $\mathbf{sSet}_Q$ and ∞-**Grpd**, with a suitable model structure over ∞-**Grpd**. Then, the following diagram is a zigzag of Quillen equivalences:

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, shift right, "\mathrm{Sing}"'] & \mathbf{sSet}_Q \arrow[l, shift right, "|\cdot|"'] \arrow[r, shift left, "\theta"] & \infty\text{-}\mathbf{Grpd}. \arrow[l, shift left, "\psi"]
\end{tikzcd}$$

Before proving the homotopy hypothesis, we need to check that θ and ψ are well-defined as functors between $\mathbf{sSet}_Q$ and ∞-**Grpd**. This fact is equivalent to the following lemma, which is proved in [Amr11], [McG20] and [Mar21].

Lemma 3.1. (i) *For every simplicial set X, $\theta(X)$ is an ∞-groupoid.*

(ii) *For every ∞-groupoid $\mathcal{C}$, $\psi(\mathcal{C})$ is a Kan complex.*

On the other hand, the model structure on ∞-**Grpd** needs to be defined. First, the following lemma ensures the compatibility of the weak equivalences of **tCat** and ∞-**Grpd**.

Lemma 3.2. [Amr11, Lemma 4.3]. *Let $F : \mathcal{C} \to \mathcal{D}$ be a map of ∞-groupoids. Then F is a weak equivalence of topological categories if and only if $\psi(F)$ is a weak equivalence in* $\mathbf{sSet}_Q$.

Then, we can define the model structure on ∞-**Grpd** as the one transferred from $\mathbf{sSet}_Q$ by the adjunction of θ and ψ. The proof of this theorem can be found in [McG20] and [Mar21].

Theorem 3.3. *The adjunction $\theta : \mathbf{sSet}_Q \rightleftarrows \infty\text{-}\mathbf{Grpd} : \psi$ induces a model structure on* ∞-**Grpd** *where:*

- *A morphism $F : \mathcal{C} \to \mathcal{D}$ of ∞-groupoids is a weak equivalence (fibration) if*

  $$\psi(F) : \psi(\mathcal{C}) \to \psi(\mathcal{D})$$

  is a weak equivalence (fibration) in $\mathbf{sSet}_Q$.
- *The cofibrations are the morphisms with the LLP with respect to any trivial fibration.*

Using that θ and ψ are well-defined and the model structure over ∞-**Grpd** compatible with the one in **tCat**, the homotopy hypothesis for the model of topological categories is equivalent to the following theorem:

Theorem 3.4 (Grothendieck homotopy hypothesis for **tCat**)**.** *The Quillen adjunction*

$$\theta : \mathbf{sSet}_Q \rightleftarrows \infty\text{-}\mathbf{Grpd} : \psi$$

is a Quillen equivalence. Therefore, there is a zigzag of Quillen equivalences between **Top** *and* ∞-**Grpd**.

Idea of the proof. The full proof can be found in [Amr11, Theorem 4.6] or [Mar21, Theorem 2.3.4]. The proof has three main steps:

1. Prove that $\mathrm{N}_{hc} \circ \mathrm{Sing} : \infty\text{-}\mathbf{Grpd} \to \mathbf{sSet}_Q$ is well-defined.
2. Show that the induced functor $\mathbf{Ho}(\infty\text{-}\mathbf{Grpd}) \to \mathbf{Ho}(\mathbf{sSet}_Q)$ is an equivalence (i.e. fully faithful and essentially surjective).
3. Using properties of $k^!$ and the previous result, prove that ψ also induces an equivalence.

□

References

[Amr11] Ilias Amrani. "Grothendieck's homotopy hypothesis". Version 1. In: (Dec. 6, 2011). arXiv: `1112.1251`.

[Amr13] Ilias Amrani. "Model structure on the category of small topological categories". In: *Journal of Homotopy and Related Structures* 10.1 (June 2013), pp. 63–70. DOI: `10.1007/s40062-013-0041-8`.

[Ber08] Julia E. Bergner. "Complete Segal spaces arising from simplicial categories". In: *Transactions of the American Mathematical Society* 361.01 (Aug. 2008), pp. 525–546. DOI: `10.1090/s0002-9947-08-04616-3`.

[Gro83] A. Grothendieck. "Pursuing Stacks". An unpublished letter to Quillen. 1983.

[JT07] André Joyal and Myles Tierney. "Quasi-categories vs Segal spaces". In: *Categories in Algebra, Geometry and Mathematical Physics*. Contemporary Mathematics 431 (2007), pp. 277–326. DOI: `10.1090/conm/431/08278`.

[Lur09] Jacob Lurie. *Higher Topos Theory (AM-170)*. Annals of Mathematics Studies 170. Princeton University Press, July 6, 2009. xviii+925. DOI: `10.1515/9781400830558`.

[Mar21] David Martínez Carpena. "Infinity groupoids as models for homotopy types". Director: Carles Casacuberta. Master's thesis. Universitat de Barcelona, Sept. 6, 2021.

[McG20] Jan McGarry Furriol. "Homotopical realizations of infinity groupoids". Director: Carles Casacuberta. Bachelor's thesis. Universitat de Barcelona, June 21, 2020. URL: `http://diposit.ub.edu/dspace/handle/2445/177842`.

[Rie11] Emily Riehl. "On the structure of simplicial categories associated to quasi-categories". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 150.3 (Mar. 2011), pp. 489–504. DOI: `10.1017/s0305004111000053`.