p-Adic model structure on simplicial sets

David Martínez Carpena

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Left Bousfield localization of model categories is one of the possible ways to create new model structures from existing ones. Under certain assumptions on the original model category, a left Bousfield localization always exists. This construction could be sum up informally as taking the same class of cofibrations but adding new weak equivalences.

The first appearance of this construction was in the context of localizing simplicial sets with respect to a generalized homology theory, by Bousfield [Bou75]. When we localize a model structure over simplicial sets with respect to a homology theory E_*, we "lose" homotopical information for every simplicial set, but retaining the E_*-accessible parts. The main examples of this type of constructions are the rational model structure and the p-adic model structure over simplicial sets.

1 Left Bousfield localization

Let $\mathcal{C}$ be a complete and cocomplete category, with a model structure (weq, cof, fib), and $S \subset \mathrm{Mor}(\mathcal{C})$. Assume that we have fibrant and cofibrant replacement functors $R, Q : \mathcal{C} \to \mathcal{C}$, and that $\mathcal{C}$ is a simplicial model category, i.e., a model category where each set of morphisms $\mathcal{C}(X, Y)$ is a simplicial set with the standard model structure.

Definition 1.1. (i) An object $Z \in \mathcal{C}$ is *S-local* if it is fibrant and for all $f : X \to Y$ in S, $f^* : \mathcal{C}(QY, Z) \to \mathcal{C}(QX, Z)$ is a weak equivalence.

(ii) A morphism $f : X \to Y$ is an *S-local equivalence* if for all S-local object $Z \in \mathcal{C}$, $f^* : \mathcal{C}(QY, Z) \to \mathcal{C}(QX, Z)$ is a weak equivalence.

(iii) A morphism $\phi : X \to X_S$ is an *S-localization* of X if X_S is S-local and ϕ is an S-local equivalence.

Proposition 1.2. *The class of S-local weak equivalences has the two-out-of-three property, is closed under retracts, and contains weq.*

Proof. See Proposition 19.4.3 of [MP12]. □

Definition 1.3. The *left Bousfield localization* $\mathcal{L}_S\mathcal{C}$ of $\mathcal{C}$ at S is, if it exists, a new model structure (weq_S, cof, fib_S) on $\mathcal{C}$ such that:

- weq_S is the class of S-local equivalences.
- $fib_S = \mathrm{rlp}(cof \cap weq_S)$.

Remark 1.4. Dually, there is a notion of right Bousfield localization defined by replacing cofibrations by fibrations and reversing directions of all arrows.

We will refer to weak equivalences of $\mathcal{L}_S\mathcal{C}$ as S-weak equivalences, to fibrations as S-fibrations, and to fibrant (resp. cofibrant) objects as S-fibrant (resp. S-cofibrant) objects. Observe that by Proposition 1.2, if $\mathcal{L}_S\mathcal{C}$ exists, it is basically an extension of the set of weak equivalences while preserving the same cofibrations.

In the following proposition we will have one of the properties depending on whether the model category is left proper. The definition of left proper can be found in A.1.

Proposition 1.5. *If $\mathcal{L}_S\mathcal{C}$ exists, the following properties hold:*

(i) Every S-fibrant object is S-local.

(ii) If $\mathcal{C}$ is left proper, then an object $X \in \mathcal{C}$ is S-local if, and only if, it is S-fibrant.

(iii) Any S-fibrant replacement $\phi : X \to R_S X$ of any object X is an S-localization of X.

Proof. See Proposition 19.2.7, Proposition 19.2.5.i and Corollary 19.2.9 of [MP12]. □

Finally, we will study certain conditions that ensure the existence of the left Bousfield localization. The condition of being a cellular model category is a technical property highly related to the proof of the following theorem. See Definition 12.1.1 of [Hir09].

Theorem 1.6. *If $\mathcal{C}$ is a left proper cellular model category, and $S \subset \mathrm{Mor}(\mathcal{C})$ is a set of morphisms, then the left Bousfield localization $\mathcal{L}_S\mathcal{C}$ exists.*

Proof. See Theorem 4.1.1 of [Hir09]. □

The proof of this theorem can be reduced to only one property of model structures. We want to prove that (weq_S, cof, fib_S) is a model structure, which is equivalent to:

(i) weq_S has the two-out-of-three property (Proposition 1.2).

(ii) All three classes are closed under retracts: It can be deduced from the other axioms of a model structure.

(iii) Liftings: We have defined fib_S such that it has the desired liftings with $cof \cap weq_S$. On the other hand, it can be easily proven that $fib_S \cap weq_S = fib \cap weq$. Then

$$cof \pitchfork fib_S \cap weq_S = cof \pitchfork fib \cap weq$$

which is true by the original model structure of $\mathcal{C}$.

(iv) Trivial fibration factorization: As before, we know that $\mathcal{L}_S\mathcal{C}$ has the same cofibrations and the same trivial fibrations, therefore this factorization is the same as in the original model structure of $\mathcal{C}$.

(v) Trivial cofibration factorization: This is the only property which is not automatic.

Therefore, the previous theorem is equivalent to proving:

Theorem 1.7. *If $\mathcal{C}$ is a left proper cellular model category, and $S \subset \mathrm{Mor}(\mathcal{C})$ is a set of morphisms, then every morphism $f : X \to Y$ factors as an S-trivial cofibration $i : X \to E_f$ followed by an S-fibration $p : E_f \to Y$.*

This is a hard problem, and in particular it also implies the existence of the S-localization for any object in $\mathcal{C}$.

2 Localizations with respect to homology theories

2.1 Existence of left Bousfield localizations

Let E_* be any homology theory on simplicial sets. We want to define a set S such that $\mathcal{L}_S$ exists and weq_S is the class of all the morphisms $f : X \to Y$ with $E_*(f) : E_*(X) \cong E_*(Y)$. First, observe that **sSet** is a left proper cellular model category, therefore this proof is a particular case of the previous existence theorem. As we have shown in the previous section, the proof of existence is equivalent to proving:

Theorem 2.1. *Every morphism $f : X \to Y$ of simplicial sets factors as an S-trivial cofibration $i : X \to E_f$ followed by an S-fibration $p : E_f \to Y$.*

The classical argument for this proof uses a very well-known categorical result:

Theorem 2.2 (Small object argument)**.** *Let $\mathcal{C}$ a locally presentable category and $I \subset \mathrm{Mor}(\mathcal{C})$ be a set of morphisms. Then every morphism $f : X \to Y$ has a factorization of the form*

$$X \xrightarrow{g} E_f \xrightarrow{h} Y$$

where $g \in \mathrm{llp}(\mathrm{rlp}(I))$ and $h \in \mathrm{rlp}(I)$. Also, $\mathrm{llp}(\mathrm{rlp}(I))$ is the set of transfinite compositions of pushouts of morphisms in I.

Proof. See Proposition 15.1.11 of [MP12]. □

A pair of simplicial sets (B, A) is a simplicial pair if there is an inclusion map $j : A \to B$ which is a cofibration. If $E_*(B, A) = 0$, then j is an S-trivial cofibration. Let k be a fixed infinite cardinal number which is at least equal to the cardinality of $E_*(*)$. For any $X \in$ **sSet** let $\#X$ denote the number of non-degenerate simplices in X.

We consider the class $\mathcal{S}$ of all the simplicial pairs (B, A) with $E_*(B, A) = 0$ and $\#B \leq k$. Thanks to the fact that each simplicial pair is a cofibration, **sSet** being cofibrantly generated (as defined in A.3), and the assumption on the number of non-degenerate simplices, we can choose a subset S of the class $\mathcal{S}$ such that every pair in $\mathcal{S}$ is isomorphic to a pair in S. Observe that this is a set of S-trivial cofibrations, and therefore a set of S-local equivalences. In particular, this is the set that will generate all S-local equivalences.

Lemma 2.3. *A morphism $p : Z \to Y$ has the RLP with respect to $S \iff p$ is S-fibration.*

Idea of the proof. Right to left implication is trivial. For left to right, it can be seen that it suffices to show that p has the RLP with respect to each simplicial pair (B, A) with $E_*(B, A) = 0$. We can obtain this result using cardinality arguments and the fact that p has the RLP with respect to S. For a complete proof, see [MP12], [GJ09] or [Bou75]. □

Thus, using the small object argument applied to $f : X \to Y$ and the set S, we obtain a factorization

$$X \xrightarrow{i} E_f \xrightarrow{p} Y$$

where $i \in \mathrm{llp}(\mathrm{rlp}(S))$ and $p \in \mathrm{rlp}(S)$. By Lemma 2.3, $\mathrm{rlp}(S) = fib_S$, and this implies that

$$\mathrm{llp}(\mathrm{rlp}(S)) = \mathrm{llp}(fib_S) = cof \cap weq_S.$$

2.2 Rational model structure on sSet

Consider the homology theory $E_* = H_*(\cdot, \mathbb{Q})$. By Theorem 2.1, we already know that this homology theory induces a left Bousfield localization. Then, take $S = H_*\mathbb{Q}$ the class of $H_*(\cdot, \mathbb{Q})$ equivalences, i.e., the morphisms $f : X \to Y$ such that $H_*(f, \mathbb{Q}) : H_*(X, \mathbb{Q}) \to H_*(Y, \mathbb{Q})$ is an isomorphism.

Theorem 2.4 (Whitehead-Serre)**.** *For X and Y simply connected simplicial sets and a morphism $f : X \to Y$, the following are equivalent:*

(i) $H_*(f, \mathbb{Q}) : H_*(X, \mathbb{Q}) \to H_*(Y, \mathbb{Q})$ *is an isomorphism.*

(ii) $\pi_*(f) \otimes \mathbb{Q} : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$ *is an isomorphism.*

Proof. See Theorem 8.6 of [FHT01]. □

Corollary 2.5. *Let X be a simply connected simplicial set. Then:*

(i) X *is rational* $\iff$ *it is a $H_*\mathbb{Q}$-fibrant object (that is a $H_*\mathbb{Q}$-local object).*

(ii) A map $\phi : X \to X_{\mathbb{Q}}$ *is the rationalization of X* $\iff$ *it is a $H_*\mathbb{Q}$-localization of X.*

2.3 p-Adic model structure on sSet

Let p be any prime and $\mathbb{F}_p$ the finite field of p elements. Consider the homology theory $E_* = H_*(\cdot, \mathbb{F}_p)$. By Theorem 2.1, we already know that this homology theory induces a left Bousfield localization. Then, take $S = H_*\mathbb{F}_p$ the class of $H_*(\cdot, \mathbb{F}_p)$ equivalences, i.e., the morphisms $f : X \to Y$ such that $H_*(f, \mathbb{F}_p) : H_*(X, \mathbb{F}_p) \to H_*(Y, \mathbb{F}_p)$ is an isomorphism.

Theorem 2.6 (Schiffman, mod p Whitehead)**.** *For X and Y simply connected simplicial sets of finite type and a morphism $f : X \to Y$ between them, the following are equivalent:*

(i) $H_*(f, \mathbb{F}_p) : H_*(X, \mathbb{F}_p) \to H_*(Y, \mathbb{F}_p)$ *is an isomorphism.*

(ii) $H_*(f, \widehat{\mathbb{Z}}_p) : H_*(X, \widehat{\mathbb{Z}}_p) \to H_*(Y, \widehat{\mathbb{Z}}_p)$ *is an isomorphism.*

(iii) $\pi_*(f) \otimes \widehat{\mathbb{Z}}_p : \pi_*(X) \otimes \widehat{\mathbb{Z}}_p \to \pi_*(Y) \otimes \widehat{\mathbb{Z}}_p$ *is an isomorphism.*

Proof. See [Sch81]. □

Definition 2.7. Let X be a simply connected simplicial set. Then:

(i) X is *p-complete* if it is a $H_*\mathbb{F}_p$-fibrant object (that is a $H_*\mathbb{F}_p$-local object).

(ii) A map $\phi : X \to X_{\mathbb{F}_p}$ is a *p-completion* of X if it is a $H_*\mathbb{F}_p$-localization of X.

Example 2.8. Every Eilenberg-Mac Lane space $K(\mathbb{F}_p, n)$ is $H_*\mathbb{F}_p$-local (since the homotopy groups of $\mathbf{sSet}(X, K(\mathbb{F}_p, n))$ can be identified with cohomology groups of X with coefficients in $\mathbb{F}_p$, and $H^*(\cdot, \mathbb{F}_p)$ equivalences coincide with $H_*(\cdot, \mathbb{F}_p)$ equivalences).

Example 2.9. For any abelian group A

$$K(A, 1)_{\mathbb{F}_p} = K(B, 1) \times K(C, 2)$$

where $B = \mathrm{Ext}(\mathbb{Z}/p^\infty, A)$ and $C = \mathrm{Hom}(\mathbb{Z}/p^\infty, A)$. If A is finitely generated, then $B = \hat{A}_p = A \otimes \widehat{\mathbb{Z}}_p$ (the p-adic completion of A) and $C = 0$, which implies $K(A, 1)_{\mathbb{F}_p} = K(A \otimes \widehat{\mathbb{Z}}_p, 1)$. The proofs can be found in [BK72] and in [MP12].

3 Application to Mandell's theorem

Let $\mathcal{E}_p$ the category of E_∞ $\overline{\mathbb{F}}_p$-algebras. Remember Mandell's main theorem [Man01]:

Theorem 3.1. *The singular cochain functor with coefficients in $\overline{\mathbb{F}}_p$ induces a contravariant equivalence from the homotopy category of connected p-complete nilpotent spaces of finite p-type to a full subcategory of the homotopy category of E_∞ $\overline{\mathbb{F}}_p$-algebras.*

Consider the singular normalized cochain functor $C^*(\cdot, \overline{\mathbb{F}}_p) : \mathbf{sSet} \to \mathcal{E}_p$. This functor over $\overline{\mathbb{F}}_p$ has the property of sending $H_*(\cdot, \mathbb{F}_p)$ equivalences to quasi-isomorphisms. The section 4 of [Man01], defines the functor $U : \mathcal{E}_p \to \mathbf{sSet}$ as right adjoint to $C^*(\cdot, \overline{\mathbb{F}}_p)$ by

$$U(E) := \mathcal{E}_p(E, C^*(\Delta[-])).$$

Using the concepts introduced in the previous sections, we can do several observations:

- In the theorem, we are considering **sSet** with the p-adic model structure and $\mathcal{E}_p$ with the semi-model structure given in the previous talk.
- The article shows that $C^*(\cdot, \overline{\mathbb{F}}_p)$ and U are Quillen adjunctions.
- The equivalence mentioned in the theorem is between subcategories of the homotopy categories. The homotopy category in the p-adic model structure over simplicial sets has all objects being $H_*\mathbb{F}_p$-fibrant and $H_*\mathbb{F}_p$-cofibrant. But all simplicial sets are $H_*\mathbb{F}_p$-cofibrant, then the homotopy category will have all the $H_*\mathbb{F}_p$-fibrant objects.
- The author restricts to connected nilpotent simplicial sets of finite p-type, because these are the spaces in which there is a general agreement about what p-complete space means. Then, under these assumptions, the $H_*\mathbb{F}_p$-fibrant simplicial sets are the p-complete simplicial sets.

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A Additional properties of model categories

During these notes, we have used several times two well-known types of model categories: left proper and cofibrantly generated. In this short appendix we want to give the definitions of these properties.

As it is well-known, a model category fibrations are stable under pullbacks, and cofibrations are stable under pushouts. But weak equivalences don't need to be stable under either construction. A left (resp. right) proper model category has the property of preserving weak equivalences under certain pushouts (resp. pullbacks).

Definition A.1. A model category is *left proper* if the weak equivalences are preserved by pushouts along cofibrations. More explicitly, for every weak equivalence $f : A \xrightarrow{\sim} B$ and every cofibrations $i : A \rightarrowtail C$ the pushout $g : C \to B \sqcup_A C$ given by the following diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow \\ C & \xrightarrow[g]{} & B \sqcup_A C \end{array}$$

is a weak equivalence.

Example A.2. Every model category in which all objects are cofibrant is left proper. In particular, **sSet** is left proper.

On the other hand, a cofibrantly generated model category can be defined in the following way:

Definition A.3. A model structure (weq, cof, fib) is *cofibrantly generated* if there exists two sets I and J such that $fib = \mathrm{rlp}(J)$ and $fib \cap weq = \mathrm{rlp}(I)$.

Example A.4. The category **sSet** with the standard model structure is cofibrantly generated by the sets:

$$I = \{\text{The boundary inclusions } \partial\Delta[n] \to \Delta[n]\}$$
$$J = \{\text{The horn inclusions } \Lambda^i[n] \to \Delta[n]\}$$