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Classifying spaces of infinity groupoids

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XXVIII ENCUENTRO DE TOPOLOGÍA

Introduction to classifying spaces

Classifying space

A *classifying space* of a topological group G is the base space BG of a G-principal bundle $EG \to BG$ such that EG is contractible.

Any classifying space is universal in the sense that for any space X, there is a bijection between homotopy classes of maps $X \to BG$ and isomorphism classes of G-principal bundles over X.

A *model* of classifying spaces is a functor from the category of topological groups to the category of G-principal bundles. The standard model for topological spaces was introduced by Milgram in 1967, and it is based on the well-known bar construction.

Similar constructions can be carried out in the setting of simplicial groups. Given a simplicial group G, the simplicial classifying space is the simplicial set with n-simplices $(WG)_n = G_{n-1} \times \cdots \times G_0$, and, after geometric realization, it is also a model of a classifying space of G.

Segal's model of classifying spaces

In 1968, Segal [1] defined a model of classifying spaces applicable to any topological category, i.e., any category enriched in topological spaces. It can be defined as a combination of two functors:

- The *bar construction* $B_\bullet(\mathcal{C})$ of a topological category $\mathcal{C}$ is the simplicial space given by

$$B_0(\mathcal{C}) := \operatorname{Obj}(\mathcal{C}) \qquad B_n(\mathcal{C}) := \overbrace{\operatorname{Mor}(\mathcal{C}) \times_{\operatorname{Obj}(\mathcal{C})} \cdots \times_{\operatorname{Obj}(\mathcal{C})} \operatorname{Mor}(\mathcal{C})}^{n}$$

  with boundaries and degeneracies presented by composition and insertion of identities respectively.
- The *topological realization* $|-|_t : \mathbf{sTop} \to \mathbf{Top}$, sending a simplicial space $X_\bullet$ to

$$|X_\bullet|_t = \int^{[n]\in\Delta} X_n \times \Delta^n.$$

Segal's model

For any topological category $\mathcal{C}$, the *Segal model* of a classifying space of $\mathcal{C}$ is

$$B\mathcal{C} := |B_\bullet(\mathcal{C})|_t\,.$$

Observe that any topological group G can be seen as a topological category $\mathcal{G}$ with only one object $*$ and $\mathcal{G}(*,*) = G$. In this case, Segal's model of $\mathcal{G}$ coincides with Milgram's model of G.

Simplicial models for topological categories

Let **sCat** denote the category of all simplicial categories, i.e., simplicially enriched categories. In this work, we study simplicial models of classifying spaces that can be generalized to simplicial categories.

Given a functor $Q : \Delta \to \mathbf{sCat}$, the simplicial Q-*nerve* $N^Q : \mathbf{sCat} \to \mathbf{sSet}$ sends $\mathcal{C} \in \mathbf{sCat}$ to the simplicial set defined for every $[n] \in \Delta$ by

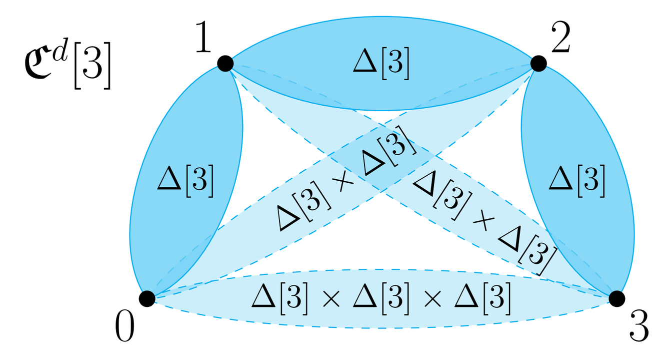
$$N^Q_n(\mathcal{C}) = \mathbf{sCat}(Q[n], \mathcal{C}).$$

Diagonal simplicial diagrams

There is a functor $\mathfrak{C}^d : \Delta \to \mathbf{sCat}$ sending $[n] \in \Delta$ to:

- $\operatorname{Obj}(\mathfrak{C}^d[n]) = \{0, \dots, n\}$.
- Morphisms of $\mathfrak{C}^d[n]$ are freely generated by the n-simplices $a_i \in \operatorname{Hom}(i-1, i)$ for all $i = 1, \dots, n$.

The $\mathfrak{C}^d$-nerve is called *diagonal nerve* $N^d : \mathbf{sCat} \to \mathbf{sSet}$.

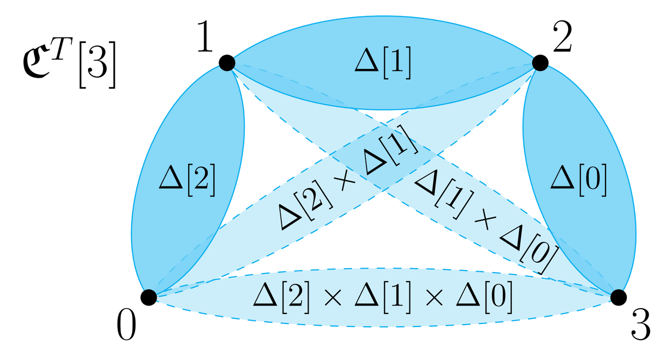


Total simplicial diagrams

There is a functor $\mathfrak{C}^T : \Delta \to \mathbf{sCat}$ sending $[n] \in \Delta$ to:

- $\operatorname{Obj}(\mathfrak{C}^T[n]) = \{0, \dots, n\}$.
- Morphisms of $\mathfrak{C}^T[n]$ are freely generated by $(n-i)$-simplices $g_i \in \operatorname{Hom}(i-1, i)$ for $i = 1, \dots, n$.

The $\mathfrak{C}^T$-nerve is called *total nerve* $N^T : \mathbf{sCat} \to \mathbf{sSet}$.



Recall that a topological category $\mathcal{C}$ is *well-pointed* if for every object $x \in \operatorname{Obj}(\mathcal{C})$ the topological monoid $\mathcal{C}(x,x)$ is well-pointed. Then, using a theorem of Cegarra and Remedios [2], we prove the following result:

Theorem

If $\mathcal{C}$ is a well-pointed topological category, then

$$|N^T \operatorname{Sing}(\mathcal{C})| \simeq |N^d \operatorname{Sing}(\mathcal{C})| \simeq |B_\bullet(\mathcal{C})|_t = B(\mathcal{C}).$$

Simplicial model for ∞-groupoids

Recall that for every topological category $\mathcal{C}$, the *homotopy category* $h\mathcal{C}$ has the same objects as $\mathcal{C}$ and $h\mathcal{C}(X,Y) := \pi_0(\mathcal{C}(X,Y))$. A topological category $\mathcal{C}$ is an ∞-*groupoid* if $h\mathcal{C}$ is a groupoid.

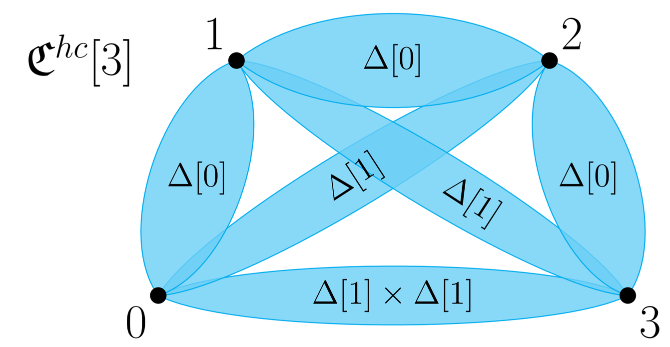
The previous two simplicial nerves do not behave well with respect to the model categorical structure of **sSet**. There is a third simplicial nerve which sets up a Quillen equivalence between **sCat** and **sSet**.

Homotopy coherent diagrams

There is a functor $\mathfrak{C}^{hc} : \Delta \to \mathbf{sCat}$ sending $[n] \in \Delta$ to:

- $\operatorname{Obj}(\mathfrak{C}^{hc}[n]) = \{0, \dots, n\}$
- For every $i, j \in \operatorname{Obj}(\mathfrak{C}^{hc}[n])$, $\operatorname{Hom}(i,j) = (\Delta[1])^{(j-i-1)}$

The $\mathfrak{C}^{hc}$-nerve is called *homotopy coherent nerve* $N^{hc} : \mathbf{sCat} \to \mathbf{sSet}$.



In our work, we show that the homotopy coherent nerve is a model of classifying spaces but only when the topological category is a well-pointed ∞-groupoid. Hence, following a similar argument as in Hinich [3], we gave in [4] a proof of the following result:

Theorem

For any well-pointed ∞-groupoid $\mathcal{X}$, the Segal model of the classifying space is weakly homotopy equivalent to the homotopy coherent model:

$$|N^{hc} \operatorname{Sing}(\mathcal{X})| \simeq |N^T \operatorname{Sing}(\mathcal{X})| \simeq |B_\bullet(\mathcal{X})|_t = B(\mathcal{X}).$$

References

[1] G. Segal. “Classifying spaces and spectral sequences”. In: *Publications mathématiques de l’IHÉS* 34.1 (Jan. 1968).

[2] A. M. Cegarra and J. Remedios. “The relationship between the diagonal and the bar constructions on a bisimplicial set”. In: *Topology and its Applications* 153.1 (Aug. 2005).

[3] V. Hinich. “Homotopy coherent nerve in Deformation theory”. In: (Apr. 19, 2007). arXiv: 0704.2503 [math.QA].

[4] D. Martínez-Carpena. “Infinity groupoids as models for homotopy types”. Director: Carles Casacuberta. Master’s thesis. Universitat de Barcelona, 2021.